Dear first year students: Attached is the final exam. You have 24 hours
to take it, however, it is really designed only to take about 3 hours (although I
may be a little off). When you are finished with the exam, please hand it in
either directly to me or to my secretary, Victoria Mason. (That is, don’t put
it in my mailbox).
1. (Effectivity functions) In the abstract social choice framework with a finite set of agents \( N \) and set of alternatives \( X \), an **effectivity function** is a correspondence mapping subsets of \( N \) (coalitions) into sets of alternatives. Formally, \( E : 2^N \setminus \emptyset \to 2^X \setminus \emptyset \) and satisfies \( E(N) = 2^X \setminus \emptyset \), and for all \( S \in 2^N \setminus \emptyset \), \( X \in E(S) \). An effectivity function generalizes the notion of a simple PAR.

Assume \( \Theta = L(X)^N \) (profiles of linear orders).

We will say that \( x \leq_{\Theta} y \) if there exists \( S \subseteq N \) and \( A \in E(S) \) for which \( x \in A \), \( y \notin A \), and for all \( z \in A, w \notin A, S \subseteq P(z, w) \). The interpretation is that, for any coalition \( S \) and any \( A \in E(S) \), \( S \) can force the social decision to lie in \( A \). Thus, \( x \) will be socially preferred to \( y \) if there exists some coalition \( S \) which would choose to force the social decision to lie in a set containing \( x \) and not containing \( y \)–they will do this if everybody in \( S \) prefers every element of \( A \) to every element outside of \( A \).

a) Show how the concept of effectivity function generalizes the concept of a simple PAR, by showing that every simple PAR can be associated with an effectivity function.

b) Give a condition which ensures that \( R(\theta) \) is well-defined for all \( \theta \) (analogous to properness for simple PAR’s).

c) Show that if an effectivity function always generates \( P \)-acyclic social preferences on \( \Theta \), then the following condition is true:

For all sequences \((S_1, \ldots, S_k)\) of coalitions and \((A_1, \ldots, A_k)\) of sets of alternatives for which for all \( j = 1, \ldots, k, A_j \in E(S_j) \), if for all \( s \neq t, A_s \cap A_t = \emptyset \), then

\[
\bigcap_{j=1}^k A_j \neq \emptyset.
\]

(Hint: this is a generalization of one direction of the Nakamura theorem—the converse is not true)

2. (Dutch books) In a Savage environment with a finite set of states of the world \( \Omega \) and outcomes space \( X = \mathbb{R} \) (monetary payoffs), a **Dutch book** is a sequence of pairs of acts \( \{(f_i, g_i)\}_{i=1}^k \) for which \( f_i \geq_R g_i \) for all \( i = 1, \ldots, k \), yet \( \sum_{i=1}^k g_i(\omega) > \sum_{i=1}^k f_i(\omega) \). Thus, a Dutch book is a sequence of bets that can be offered to a decision maker which leaves the decision maker with strictly less money, no matter what state obtains. Sometimes such a decision maker is referred to as a “money-pump.”

a) Suppose we define \( p(A) \) (the “probability” of \( A \)) to be that amount of money that a decision maker would pay for a bet on \( A \). That is, \( (p(A) 1_\Omega) I (1_A) \). (here, \( 1_A(\omega) = 1 \) if \( \omega \in A \), 0 if \( \omega \notin A \)). Suppose further that the decision maker has homothetic preferences, so that for all \( \lambda > 0, (\lambda p(A) 1_\Omega) I (\lambda 1_A) \). Show that if the function \( p \) is not additive, then there exists a Dutch book. (If this is too difficult, try to show it for \( \Omega \) with only two states).

b) Decision makers in the Ellsberg paradox typically will not have an additive function \( p \), and there will thus exist a “Dutch book” against them. Do you think that this implies that anyone exhibiting Ellsberg-type behavior can be used to generate an infinite amount of money, or is there something else going on here?
3. For $|X| = 2$, characterize the class of PAR’s satisfying anonymity and neutrality on the domain $\Theta = L(X)^N$. 