1. Kreps, Chapter 6, #11

2. A (simple) lottery \( p \) on \( \mathbb{R}_+ \) is said to first order stochastically dominate a simple lottery \( q \) if for all \( x \in \mathbb{R}_+ \), \( p(\{y : y \geq x\}) \geq q(\{y : y \geq x\}) \).
   
   a) Prove that the binary relation "first order stochastically dominates" is not complete.
   
   b) Prove that if \( p \) first order stochastically dominates \( q \) then for all \( u : \mathbb{R}_+ \to \mathbb{R} \) which are weakly increasing, (weakly increasing means \( x \geq y \implies u(x) \geq u(y) \)),
   \[ E_p[u] \geq E_q[u]. \]

   c) Now prove that if for all \( u : \mathbb{R}_+ \to \mathbb{R} \) which are weakly increasing,
   \[ E_p[u] \geq E_q[u], \]
   then \( p \) first order stochastically dominates \( q \) (Hint: for all \( x \in \mathbb{R}_+ \), you should find some \( u \in \mathbb{R}_+ \) such that \( E_p[u_x] = p(\{y : y \geq x\}) \)).

3. Given \( \{p_i\}_{i=1}^K \subset \Delta(X) \) and \( \{q_i\}_{i=1}^K \) for which \( p_i P q_i \) for all \( i = 1, \ldots, m \leq K \), and \( p_i R q_i \) for all \( i = 1, \ldots, K \), we will say that \( (p_i, q_i)_{i=1}^K \) refutes the expected utility hypothesis if there does not exist \( u : X \to \mathbb{R} \) for which for all \( i \),
   \[ p_i R q_i \implies u \cdot p_i \geq u \cdot q_i, \]
   \[ p_i P q_i \implies u \cdot p_i > u \cdot q_i. \]

   \( (p_i, q_i)_{i=1}^K \) is a finite (able to be determined by experiment) list of data on preferences.
   
   a) Show that \( (p_i, q_i) \) refutes the expected utility hypothesis if and only if for all \( i = 1, \ldots, K \), there exists \( \lambda_i \geq 0 \), where for some \( i \in \{1, \ldots, m\} \), \( \lambda_i > 0 \), for which
   \[ \sum_{i=1}^K \lambda_i p_i = \sum_{i=1}^K \lambda_i q_i. \]

   b) Explain how this is a type of “generalized” independence axiom (that is, show that if we have data satisfying the hypotheses of the independence axiom, then the conclusion of the independence axiom must hold if the data does not refute the expected utility hypothesis).

   c) What if \( X = \mathbb{R} \) and we would like to require that \( x > y \implies u_x > u_y \)?
   Can this model be refuted? Provide some suggestions...

4. Prove that if \( X \) is finite and \( U : \Delta(X) \to \mathbb{R} \) satisfies \( U(\alpha p + (1-\alpha) q) = \alpha U(p) + (1-\alpha) U(q) \) for all \( \alpha \in (0, 1) \), \( p, q \in \Delta(X) \), then for all \( p \),
   \[ U(p) = \sum_{x \in X} p(x) U(\delta_x). \]
   
   (Hint: this is a simple induction argument).