PS/Ec 172
Final Exam

Please take at most four hours to complete this exam. Collaboration is not allowed, except on the bonus question, which you can discuss with others, given that this discussion takes place in public conversations on Piazza. Also, there is no time limit for completing the bonus question, and you can read it (on this page, below) before starting the four hour clock.

You may consult the lecture notes, your own notes or any textbook, but not solutions to previous exams. There is no need to prove any statements that you make.

The exam is due at 1pm on June 9$^{\text{th}}$ for seniors, and 1pm on June 16th for the non-seniors.

Bonus question. Choose an integer between 0 and 10, equal to the number of bonus points you would like to have. If the average number of points asked by the students taking this exam is strictly less than 4, then you will get the number of points you asked for. Otherwise you will get no points.

Note: bonus points will be added *after* any curving. Averages will be taken separately for seniors and non-seniors.

Omer Tamuz. Email: tamuz@caltech.edu.

(1) A class of n students takes an exam. In one of the questions the students are instructed to write an integer between 0 and 10. That is, each student i writes a number $x_i \in \{0, 1, 2, \ldots, 10\}$. The numbers $\{x_i\}_{i=1}^n$ that the students write are collected, and the average $a = \frac{1}{n}\sum_i x_i$ is calculated. If $a < 4$, then each student i gets x_i points for that question. Otherwise all the students get no points for that question.

(a) Assume that $n = 100$, and that all students submit their exams simultaneously and without observing the others' answers. We say that a strategy profile is symmetric if all students play the same strategy. That is, $(x_1, \ldots, x_n)$ is symmetric if $x_i = x_j$ for all i, j.

(i) *5 points.* Which (if any) of the 11 symmetric strategy profiles are equilibria?

(ii) *5 points.* Find a (non-symmetric) pure equilibrium in which some of the students choose 3 and some choose 4.

(iii) *5 points.* Which (if any) of the strategies $\{0, \ldots, 10\}$ is weakly dominated? Which (if any) is strictly dominated?

(b) *15 points.* Assume now that $n = 2$, and that, as before, both students submit their exams simultaneously and without observing the others' answers. Consider the mixed strategy profile in which both student 1 and student 2 choose 3 with probability p and choose 4 with probability $1 - p$. For which p is this a mixed Nash equilibrium?

(c) *15 points.* Assume again that $n = 2$, but that now student 1 submits the exam before student 2, and that student 2 knows what student 1 submitted before making her choice; this is thus an extensive form game with perfect information in which student 1 plays before student 2. Find all pure Nash equilibria of this game, and state which ones are subgame perfect.

(d) Assume again that $n = 2$. The students complete the exam without observing each other's choices. When they get their exams back they learn whether they got points for that question.
Formally, the students' knowledge space after getting their exam back is $\Omega = \{0, \ldots, 10\}^2$. At $(x_1, x_2) \in \Omega$ student 1 knows x_1, and student 2 knows x_2, and additionally both know whether or not $(x_1 + x_2)/2 < 4$. We will say that a student was greedy if she chose 4 or higher. We will say that a student feels awkward if she knows that the other student knows that she was greedy.
Assume that $x_1 = 3$ and $x_2 = 5$.

(i) *5 points.* What is the partition element $P_1(3, 5)$ of student 1? That is, what is the set of values of the pair (x_1, x_2) that is consistent with the information that she knows?

(ii) *5 points.* What is the partition element $P_2(3, 5)$ of student 2?

(iii) *5 points.* Does student 1 know that student 2 was greedy?

(iv) *5 points.* Does student 2 feel awkward?

(v) *5 points.* Does student 2 know that student 1 was not greedy?

(2) Consider the following extensive form game with incomplete information, played by a student, a teacher and an employer.

- The student decides whether or not to study.
- Studying costs 1/3, while not studying costs nothing.
- If she studies then she will be able to solve the exam.
- If she does not study then she will be able to solve the exam with probability $0 < p < 1$.
- She takes the exam, and solves it if she can.
- The teacher decides whether to read the exam.
- The teacher does not know if the student studied.
- Reading the exam costs 1/4.
- If the teacher reads the exam he knows whether it was solved.
- The teacher has to give a grade: either pass or fail.
- The teacher gets utility 1 from passing a solved exam or failing an exam that was not solved. Otherwise he gets utility 0.
- The employer observes the grade, and gets utility 1 from hiring a student who can solve the exam, utility -1 from hiring a student who cannot, and utility of 0 from not hiring.
- The student gets utility 1 from getting hired, and utility 0 from not getting hired.

In summary: p is the probability that the student can solve the exam without stuyding. The student has to decide whether or not to study. The teacher has to decide whether or not to read the exam, and then has to grade it; she does not know if the student studied, and needs to read the exam to know if it was solved. The employer, who observes the grade, has to decide whether or not to hire. The questions below are regarding **pure** strategies.

The following questions are for 10 points each.

(a) **Harvard.** For which values of p does there exist an equilibrium in which the student **does not** study, the teacher **does not** read the exam, the teacher **passes** the student and the employer **hires** the student?

(b) **Berkeley.** For which values of p does there exist an equilibrium in which the student **does not** study, the teacher **does** read the exam, the teacher **either passes or fails** the student (depending on how she did on the exam), and the employer **hires** the student if and only if she passed?

(c) **Caltech.** For which values of p does there exist an equilibrium in which the student **does** study, the teacher **does not** read the exam, the teacher **passes** the student and the employer **hires** the student?