MA144A, HOMEWORK 7
DUE FRIDAY, DECEMBER 4TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let X be a bounded random variable with mean 0. Let $X, X_1, X_2, \ldots$ be an i.i.d. sequence, and let $Z_n = \frac{1}{n}\sum_{k=1}^n X_k$.

(a) Show that conditioned on a large deviation, the deviation is as small as possible, in the sense that for every $\eta > 0$ with $\mathbb{P}[X \geq \eta] > 0$, and for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[\eta \leq Z_n \leq \eta + \varepsilon | \eta \leq Z_n] = 1.$$

(b) Suppose X is symmetric: $\mathbb{P}[X \leq a] = \mathbb{P}[-X \leq a]$ for all $a \in \mathbb{R}$.
Let $\eta > 0 > \zeta$ with $\mathbb{P}[X \geq \eta] > 0$ and $\mathbb{P}[X \leq \zeta] > 0$. Let A_n be the event $\{Z_n \geq \eta \text{ or } Z_n \leq \zeta\}$. Calculate the rate $\lim_n -\frac{1}{n}\log \mathbb{P}[A_n]$ in terms of $K_X^\star$, η and ζ.

(2) Let G be a group generated by the finite, symmetric set S of size d. The transition matrix P of the associated simple random walk is given by

$$P(g,h) = \begin{cases} \frac{1}{d} & \text{if } h = gs \text{ for some } s \in S \\ 0 & \text{otherwise.} \end{cases}$$

Note that $P(g,h) = P(h,g)$, since S is symmetric. For $f \colon G \to \mathbb{R}$ we denote by Pf the function on G given by $[Pf][g] = \sum_h P(g,h)f(h) = \sum_h P(h,g)f(h)$. Note that $f \mapsto Pf$ is a linear operator.

For $f \colon G \to \mathbb{R}$ We denote $\|f\|_p = \left(\sum_g |f(g)|^p\right)^{1/p}$ and $\ell^p(G) = \{f \colon G \to \mathbb{R} \,:\, \|f\|_p < \infty\}$.

Given a subset $F \subseteq G$, we denote

$$\partial F = \{g \in F \,:\, gs \notin F \text{ for some } s \in S\}.$$

Omer Tamuz. Email: tamuz@caltech.edu.

We say that G is *amenable* if

$$\inf_{\text{finite} F \subset G} \frac{|\partial F|}{|F|} = 0$$

(If you are interested, you can verify that amenability does not depend on the choice of generating set.)

(a) Prove that if $f \geq 0$ then $\|Pf\|_1 = \|f\|_1$.

(b) Prove that if $f \geq 0$ then $\|Pf\|_2 \leq \|f\|_2$.
Hint: Write $P = \frac{1}{d}(P_1 + \cdots + P_d)$, where for each P_i there is an $s_i \in S$ such that $P_i(g,h) = 1$ iff $h = gs_i$. Then use the fact (which follows from the triangle inequality) that $\| \sum_i \alpha_i f_i \|_2 \leq \sum_i |\alpha_i| \cdot \|f_i\|_2$.

(c) Prove that if G is amenable then for each $\varepsilon > 0$ there is an $f \in \ell^2(G)$ such that $\|Pf\|_2 \geq (1-\varepsilon)\|f\|_2$.

(d) Kesten's Theorem states that if G is nonamenable then there exists an $r < 1$ such that $\|Pf\|_2 < r\|f\|_2$. Use this theorem to prove that a simple random walk on a nonamenable group has positive random walk entropy.

(3) **For your own amusement (please do not submit).** The *lamplighter group* $LL(\mathbb{Z})$ is defined as follows. It is the set of pairs (A, z), where A is a finite subset of $\mathbb{Z}$, and z is an element of Z. The group operation is $(A,z)\cdot(B,w) = (A \triangle (B+z), z+w)$.

(a) Prove that $LL(\mathbb{Z})$ is indeed a group.

(b) Prove that $LL(\mathbb{Z})$ is generated by $S = \{s_1, s_2, s_3\}$ where $s_1 = (\{0\}, 0)$, $s_2 = (\emptyset, 1)$ and $s_3 = (\emptyset, -1)$.

(c) Prove that $LL(\mathbb{Z})$ is amenable.

(d) Find a random walk supported on $S = \{s_1, s_2, s_3\}$ (not necessarily with the uniform distribution) that has positive random walk entropy.