MA140A, HOMEWORK 2
DUE FRIDAY, OCTOBER 16^{TH}

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

Recall that by the *Prime Number Theorem*, the probability that a number chosen uniformly from $\{1, 2, \ldots, n\}$ is prime is of order $1/\log n$. Formally, if Y_n is distributed uniformly on $\{1, \ldots, n\}$, and if $\mathcal{P}$ denotes the set of primes, then

$$\lim_n \frac{\mathbb{P}[Y_n \in \mathcal{P}]}{1/\log n} = 1.$$

Inspired by this result (due to Jacques Hadamard and Charles Jean de la Vallée Poussin), we will choose a *random* subset P of the natural numbers that will resemble the primes in the sense described above.

Let $(X_3, X_4, \ldots)$ be a sequence of independent random variables taking values in $\{0, 1\}$, whose distributions are given by $\mathbb{P}[X_n = 1] = p_n$ and $\mathbb{P}[X_n = 0] = 1 - p_n$, with

$$p_n = \frac{n+1}{\log(n+1)} - \frac{n}{\log n}$$

Let P be a random variable taking values in the space of subsets of $\mathbb{N}$ (which can be identified with $\{0, 1\}^{\mathbb{N}}$), and given by

$$P = \{n \,:\, X_n = 1\}.$$

Recall that the (unproven) Goldbach conjecture states that every even number greater than 2 is the sum of two primes. A weaker (and still unproven) conjecture is that there is some N so that every even $n \geq N$ is the sum of two primes. We will show that our random P will satisfy an analogue of this conjecture.

(1) Let $Y_1, Y_2, \ldots$ be a sequence of random variables, each independent of P, and such that $\mathbb{P}[Y_n = i] = 1/n$ if $1 \leq i \leq n$ and $\mathbb{P}[Y_n = i] = 0$ otherwise. Prove that

$$\lim_n \frac{\mathbb{P}[Y_n \in P]}{1/\log n} = 1.$$

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(2) Let G_n be the event that n is not the sum of any two elements of P. Show that $\mathbb{P}[G_n] < n^{-2}$ for all n large enough.[1]

(3) Let G be the event that there exists some integer K such that every $n \geq K$ is the sum of two elements of P. Using the Borel-Cantelli Lemma, show that $\mathbb{P}[G] = 1$.

[1] In fact, $\mathbb{P}[G_n]$ decreases much more rapidly than that, but this bound will suffice for our needs.