MA140A, HOMEWORK 1
DUE FRIDAY, OCTOBER 9TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let I be a set, and let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of sigma-algebras of subsets of Ω. Show that $\cap_{i \in I} \mathcal{F}_i$ is a sigma-algebra.

(2) Let $\mathcal{C}$ be a collection of subsets of Ω. Show that there exists a unique minimal (under inclusion) sigma-algebra $\mathcal{F} \supseteq \mathcal{C}$.

(3) Given $n, k \in \mathbb{N} = \{1, 2, \ldots\}$, denote by $A_{n,k} = \{nz + k \,:\, z \in \mathbb{Z}\}$ the set of all integers that are equal to k mod n. Let $\mathcal{P} = \{A_{n,k} \,:\, n, k \in \mathbb{N}\} \cup \{\emptyset\}$. Let $\mathcal{A}$ be the collection of finite unions of elements of $\mathcal{P}$.
  (a) Show that $\mathcal{P}$ is a π-system and that $\mathcal{A}$ is an algebra.
  (b) Find a finitely additive probability measure on $\mathcal{A}$.
  (c) Is $\mathcal{A}$ a sigma-algebra?
  (d) **Bonus.** Is your measure sigma-additive?

(4) Let $\Omega = \mathbb{Z}$ and $\mathcal{F} = 2^{\mathbb{Z}}$ be the power set of $\mathbb{Z}$. A finitely additive probability measure $\mu \colon \mathcal{F} \to [0,1]$ is *shift-invariant* if for all $A \in \mathcal{F}$ it holds that $\mu(A) = \mu(A+1)$, where

$$A + 1 = \{n + 1 \,:\, n \in A\}.$$

  (a) Prove that if μ is shift-invariant then it is not sigma-additive.
  (b) *Bonus.* Prove that there exists such a shift-invariant μ.

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