18.022 2014, Homework 8. Due Thursday, November 6th

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth six points. Three additional points will be given to any assignment in which there is an honest attempt to answer every question.

(1) Let A be an open subset of $\mathbb{R}^2$, let $f\colon A \to \mathbb{R}$ in $\mathcal{C}^2$ be harmonic, and let the determinant of the Hessian Hf be non-zero everywhere. We will show in this problem that f has no local maxima or minima. This is called the *maximum principle.*

(a) For a given $x \in A$, explain why the Hessian $Hf(x)$ is of the form
$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$
for some $a, b \in \mathbb{R}$.

(b) Explain why any given $x \in A$ is not a local maximum or minimum by using Proposition 12 in lecture 20.

(c) Let
$$S = \{(x, y) \in \mathbb{R}^2 \,:\, 0 \le x \le 1, 0 \le y \le 1\}.$$
Let $f\colon S \to \mathbb{R}$ in $\mathcal{C}^2$ be harmonic, and let the determinant of its Hessian be non-zero everywhere. What is the boundary of S? Explain why f attains a maximum and a minimum on the boundary of S.

(2) The Konman & Schwindler Consulting Company was hired to help price the new iSmell© online smell transmitter, which costs c dollars to make. Their research yielded a function $f\colon [0, \infty] \to [0, 1]$ such that $f(x)$ is the fraction of potential customers to whom an iSmell© is worth at least x dollars. For example, if $f(250) = 0.9$ then to 90% of the potential customers an iSmell© is worth $250 or more. Assume that if an iSmell© is priced at x than anyone to whom it is worth x or more will buy it.

Assume that f is $\mathcal{C}^2$ and that $f'(x)$ is never equal to 0.

(a) Let there be N potential customers, and let $P(x)$ be the amount of money that the manufacturer will make (the manufacturer's profit) if it sets the price at x dollars. Calculate $P(x)$. (Note: don't worry about rounding customer numbers to the nearest whole number.)

(b) Calculate P' to write an equation satisfied by any price $x_m \in (0, \infty)$ which is a critical point of P. In this equation, express x_m as a function of c, $f(x_m)$ and $f'(x_m)$. What inequality involving P'' can be used to show that a price will maximize profits?

(c) Assume now that $f\colon [0 \to \infty)$ is given by $f(x) = e^{-x/a}$ for some $a > 0$. As a function of a and c, what price in $[0, \infty)$ should Konman & Schwindler advise the manufacturer to set? Make sure you check both interior and boundary points.

(3) We will show in this problem that among all boxes with surface area a, the volume is a maximum if and only if the box is a cube. Let
$$V\colon \mathbb{R}^3 \to \mathbb{R},$$
be the function $V(x, y, z) = xyz$. Then we want to maximize V on the set A of all points where $x > 0$, $y > 0$, $z > 0$ and $2(xy + yz + zx) = a$.

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Let $K \subset A$ be the subset of points (x, y, z) where

$$x \geq \frac{\sqrt{a}}{3\sqrt{6}} \qquad y \geq \frac{\sqrt{a}}{3\sqrt{6}} \qquad \text{and} \qquad z \geq \frac{\sqrt{a}}{3\sqrt{6}}.$$

(a) Show that there is a unique point $p \in A$ where V has a critical point.
(b) Show that if $q \in A \setminus K$ (so that q is in A but not K) then $V(q) < V(p)$.
(c) Show that K is bounded.
(d) Show that K is closed by showing that if $p_n = (x_n, y_n, z_n)$ is a sequence of points in K with limit $p = (x, y, z)$, then $(x, y, z) \in K$. (Hint: use the fact that the function $S(x, y, z) = 2(xy + yz + xz)$ is continuous.)
(e) Show that V has a maximum on K at p.
(f) Show that V has a maximum on A at p.

(4) Let $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^2$. Let $h(x, y) = f(x)g(y)$. Let x_0 satisfy $f'(x_0) = 0$ and $f''(x_0) < 0$ (so that x_0 is a local maximum of f) and likewise let y_0 satisfy $g'(y_0) = 0$ and $g''(y_0) < 0$. Let $f(x_0) = a_0 \neq 0$, and let $g(y_0) = b_0 \neq 0$.
(a) Show that $p_0 = (x_0, y_0)$ is a critical point of h.
(b) Let A be an n-by-n diagonal matrix, and let $Q \colon \mathbb{R}^n \to \mathbb{R}$ be given by $Q(\vec{x}) = \vec{x}^t A \vec{x}$. How can one simply determine if Q is positive or negative definite, given A?
(c) Show that $p_0 = (x_0, y_0)$ is a local maximum, a local minimum or a saddle point of h, depending on a_0 and b_0. (Hint: use proposition 12 in lecture 20.)

(5) Recall that Ruth and Bernie M. need to decide how to invest their (and other people's) money. Suppose that there are n available stocks, and suppose that the Konman & Schwindler Company provides them with a number x_i for every stock, which is the expected profit to be made by investing in that stock, as a fraction of the amount of money invested. Let p_i be the amount of money invested in each stock. For example, if $x_1 = 0.01$ then investing $p_1 = \$700$ in stock 1 is expected to yield $0.01 \cdot \$700 = \$7$ in profit. Note that x_i can be negative, as can p_i - this corresponds to buying a negative number of stocks, also called "shorting the stock".

The risk of a portfolio $p = (p_1, \ldots, p_2)$ is given by $\sigma(p) = |p| = \sqrt{\sum_i p_i^2}$. Ruth and Bernie are required by the SEC (the Securities and Exchange Commission is the US federal regulatory body overseeing financial service providers) to have a portfolio p with $\sigma(p)$ no more than $\$1,000,000,000$.
(a) Calculate $e(p)$, Ruth and Bernie's expected total profit for a given portfolio p, given x. Assuming that it is possible for them to make non-zero profit, explain why they will choose to have $\sigma(p) = \$1,000,000,000$ (that is, the largest they are allowed to). (Hint: assume that some p maximizes the profit, and show that if $\sigma(p) \leq \$1,000,000,000$ then $\sigma(p) = \$1,000,000,000$.) Explain why Ruth and Bernie have an incentive to cheat and increase the risk of their portfolio.
(b) Calculate the portfolio p that maximizes $e(p)$ among all legal portfolios, and the expected profit at this portfolio. (Hint: you can constrain $\sigma(p)^2$ instead of $\sigma(p)$.)