18.022 2014, Homework 6. Due Thursday, October 23RD

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth six points. Three additional points will be given to any assignment in which there is an honest attempt to answer every question.

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^2$. Let

$$S = \{(x, y) \in \mathbb{R}^2 \, : \, y = f(x)\}$$

be the graph of f. Where appropriate, express the answers to the questions below in terms of f and its derivatives.

- (a) Find a parametrized differentiable curve $\vec{r}\colon \mathbb{R} \to \mathbb{R}^2$ of class $\mathcal{C}^2$ whose image (i.e., the set of points $\vec{v} \in \mathbb{R}^2$ such that $\vec{v} = \vec{r}(t)$ for some $t \in \mathbb{R}$) is S. Calculate its derivative and its second order partial derivatives, and explain why it is indeed of class $\mathcal{C}^2$.
- (b) Calculate $s(t)$, the arclength parameter of $\vec{r}$, relative to the reference point $t = 0$. Use this to calculate the length of $S \cap [0, 1]$ (the graph of f restricted to the interval $[0, 1]$).
- (c) Calculate the unit tangent vector $\vec{T}(t)$.
- (d) Calculate the unit normal vector $\vec{N}(t)$ and the curvature $\kappa(t)$ of $\vec{r}$. Explain what condition f must satisfy at a point $t \in \mathbb{R}$ for these to be well defined.

(2) Let a, b be positive constants. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^2$ functions with the property that $a^2 g'(t)^2 + b^2 f'(t)^2 = 1$ (for example, $g(t) = \frac{1}{a} \cos t$ and $f(t) = \frac{1}{b} \sin t$ satisfy this).

Let $\vec{r}\colon \mathbb{R} \to \mathbb{R}^3$ be given by $\vec{r}(t) = (a \cos g(t), a \sin g(t), b f(t))$.

Where appropriate, express the answers to the questions below in terms of f, g, and their derivatives.

- (a) Write $\vec{r}(t)$ in cylindrical coordinates, and explain why this curve lies on the cylinder of radius a.
- (b) Recall the orthonormal cylindrical coordinate frame $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ at a point $(r, \theta, z) \in \mathbb{R}^3$. These depend only on θ, and so can think of each as a parametrized curve, given as a function of θ.

  Explain why $\frac{d\hat{e}_r(\theta)}{d\theta} = \hat{e}_\theta(\theta)$, $\frac{d\hat{e}_\theta(\theta)}{d\theta} = -\hat{e}_r(\theta)$ and $\frac{d\hat{e}_z(\theta)}{d\theta} = \vec{0}$.
- (c) Express $\vec{r}(t)$ and $\vec{r}'(t)$ in terms of the cylindrical coordinate frame vectors.
- (d) Calculate the arclength parameter with respect to the reference point $t = 0$.
- (e) Calculate the unit tangent vector $\vec{T}(s)$.
- (f) Express the curvature $\kappa(t)$ in terms of the cylindrical coordinate frame vectors. Use this to calculate the curvature of

$$\vec{x}(t) = \left(\frac{\sqrt{2}}{2} \cos(\sin t), \frac{\sqrt{2}}{2} \sin(\sin t), \frac{\sqrt{2}}{2} \cos(t) \right).$$

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(3) Let $\vec{v}\colon I \to \mathbb{R}^n$ be a continuous parametrized curve, and denote $\vec{v}(t) = (v_1(t), v_2(t), \ldots, v_n(t))$. Let $a \in I$, and assume that for every $t \in I$ and i between 1 and n the integral

$$\int_a^t v_i(\tau)d\tau$$

exists and is finite.

Let $x \in \mathbb{R}^n$, and let $\vec{r}\colon I \to \mathbb{R}^n$ be given by

$$r_i(t) = x_i + \int_a^t v_i(\tau)d\tau.$$

(a) Using the fundamental theorem of calculus, explain why $\vec{r}$ is a differentiable parametrized curve with $\vec{r}\,' = \vec{v}$. (Hint: recall that the fundamental theorem of calculus states that if $f\colon I \to \mathbb{R}$ is continuous and $F\colon I \to \mathbb{R}$ is given by $F(x) = b + \int_a^x f(t)dt$ for some $b \in \mathbb{R}$ and $a \in I$, then F is differentiable on I and $F'(x) = f(x)$.)

(b) Let $\vec{r}_1\colon \mathbb{R} \to \mathbb{R}$ and $\vec{r}_2\colon \mathbb{R} \to \mathbb{R}$ be two smooth (i.e., $\mathcal{C}^\infty$) curves such that $\vec{r}_1(a) = \vec{r}_2(a)$ for some $a \in \mathbb{R}$, and $\vec{r}_1\,'(t) = \vec{r}_2\,'(t)$ for all $t \in \mathbb{R}$. Explain why $\vec{r}_1(t) = \vec{r}_2(t)$ for all $t \in \mathbb{R}$.

(4) In this question, we prove the uniqueness statement of Theorem 2.5 on page 201 of the book. Let $\vec{r}_1\colon I \to \mathbb{R}^3$ and $\vec{r}_2\colon I \to \mathbb{R}^3$. be two smooth regular curves parametrised by arclength. Assume that $\kappa_1(s) = \kappa_2(s)$ and $\tau_1(s) = \tau_2(s)$, for every $s \in I$ (and so call them just $\kappa(s)$ and $\tau(s)$). Suppose that there is a point $a \in I$ where

$$\vec{r}_1(a) = \vec{r}_2(a), \quad \vec{T}_1(a) = \vec{T}_2(a), \quad \vec{N}_1(a) = \vec{N}_2(a), \quad \text{and} \quad \vec{B}_1(a) = \vec{B}_2(a).$$

(a) Show that the quantity

$$f(s) = \|\vec{T}_1(s) - \vec{T}_2(s)\|^2 + \|\vec{N}_1(s) - \vec{N}_2(s)\|^2 + \|\vec{B}_1(s) - \vec{B}_2(s)\|^2,$$

is a constant function of s. (Hint: Differentiate and use the Frenet-Serret formulae.)

(b) Explain why $\vec{T}_1(s) = \vec{T}_2(s)$, $\vec{N}_1(s) = \vec{N}_2(s)$ and $\vec{B}_1(s) = \vec{B}_2(s)$, for all $s \in S$.

(c) Explain why $\vec{r}_1(s) = \vec{r}_2(s)$, for all $s \in S$. (Hint: use question 3.)

(5) Let a,b and c be positive constants, with $c^2 = a^2 + b^2$.

(a) Let $\vec{r}\colon \mathbb{R} \to \mathbb{R}^3$ be given by

$$\vec{r}(t) = \left(a \cos \frac{t^3}{c}, a \sin \frac{t^3}{c}, \frac{bt^3}{c}\right).$$

Calculate the curvature $\kappa(t)$ and the torsion $\tau(t)$. (Hint: find a way to use what was proved in Example 8 of lecture 15.)

(b) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $g(0) = 0$, and now define

$$\vec{r}(t) = \left(a \cos \frac{g(t)}{c}, a \sin \frac{g(t)}{c}, \frac{bg(t)}{c}\right).$$

Calculate $\kappa(t)$ and $\tau(t)$, and give a geometrical explanation to what is going on.