18.022 2014, Homework 5. Due Thursday, October 16^{TH}

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth six points. Three additional points will be given to any assignment in which there is an honest attempt to answer every question.

(1) A GPS system is comprised of some number of satellites $\{S_1, \dots, S_n\}$ orbiting the earth. The satellites have synchronized clocks, and each second they all, at the same time, emit a signal. The signal emitted by S_i contains $\vec{v}_i \in \mathbb{R}^3$, the position of S_i, and travels at the speed of light c.

Commuter Joe has a GPS antenna and receives the signals emitted by the satellites at some time t. He therefore knows the satellites positions $\vec{v}_1, \dots, \vec{v}_n$ at that time. He also has a clock that records the times $t_1, \dots, t_n$ at which the signals arrived at his antenna. However, he does not know the time t, since his clock is not synchronized with that of the satellites.

(a) Express d_i, the distance from Joe to S_i, in terms of t, t_i and c. Note that Joe does not know t, and therefore cannot calculate this distance.

(b) Given any two satellites S_i and S_j, express the difference $d_i - d_j$ in terms of quantities that are known to Joe (these include the speed of light c).

(c) Assume now that Joe knows that he is on the x-y-plane, so that he is at some $x = (x_1, x_2, 0)$. Assume also that there are only two satellites, and that he learned that $\vec{v}_1 = (-500, 0, 0)$, that $\vec{v}_2 = (500, 0, 0)$, and that $d_1 - d_2 = 800$. What is the set of points that Joe conceivably could be on? Write an equation that is satisfied by x_1 and x_2 on this set of points, draw a rough sketch, and explain why this is one branch of a hyperbola.

(d) Assume still that Joe is on the x-y plane. This time, however, there are three satellites, with some positions $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which are all on the same plane. We will think of Joe's position x, as well as these three vectors, as elements of $\mathbb{R}^2$.
Joe would like to calculate his position. Write a set of three equations involving c, x, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and t_1, t_2, t_3; there should be one equation for each pair of satellites. Explain why the third equation can be derived from the first two.

(e) Solving the system of equations from the previous question can be hard. Instead, Joe first defines a function $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x) = (0, 0)$ if and only if the system of equations is satisfied (note that only the first two equation are important, since the third follows from them). He then uses Newton's method to find an approximate root of f, which will be an approximate solution of the set of equations.
Write the function f, and explain how to use Newton's method here. (Note: there's no need to explicitly calculate the inverse of Df.)

(2) Elliptic coordinates on the plane are given by two numbers (μ, ν) such that $\mu \geq 0$ and $\nu \in [0, 2\pi)$, and where

$$x = \cosh(\mu)\cos(\nu)$$

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and

$$y = \sinh(\mu)\sin(\nu),$$

Recall that this would be an *orthogonal coordinate system* if the vectors $(\partial\mu/\partial x, \partial\mu/\partial y)$ and $(\partial\nu/dx, \partial\nu/\partial y)$ were orthogonal.

(a) Explain why this coordinate system is called elliptic, and why it could perhaps be more appropriately named elliptic-hyperbolic.

(b) Let f be a function that assigns to each point in the plane a real number; we can think of it as either a function of (x, y) or as a function of (μ, ν). For $\nu \neq 0$, Calculate the partial derivatives $\partial f/\partial\mu$ and $\partial f/\partial\nu$ in terms of $\partial f/\partial x$ and $\partial f/\partial y$, and vice versa.

(c) Explain why the elliptic coordinate system is orthogonal. (Hint: use the previous problem to calculate the relevant partial derivatives.)

(3) A mirror M is placed on the x-y-plane in $\mathbb{R}^3$. A ray of light traveling in direction $\hat{r} = (r_1, r_2, r_3)$ (a unit vector in $\mathbb{R}^3$) hits M at a point $x = (x_1, x_2, 0)$. The rules of optics tell us that it will be reflected in direction $(r_1, r_2, -r_3)$.

More generally, if a ray is reflected off a plane with normal $\vec{n}$, then its new direction will be $f(\hat{r}) = \hat{r} - 2\mathrm{proj}_{\vec{n}}\hat{r}$. Even more generally, if a ray of light is reflected off a surface S at a point in which $\vec{n}$ is the normal to the tangent plane of S, then it will be reflected in the direction $f(\hat{r})$.

(a) Explain why in the first case (that the ray is reflected off the x-y-plane) applying the more general rule yields the same answer, $(r_1, r_2, -r_3)$.

(b) Draw a diagram that makes it clear that the general rule is indeed a generalization of the rule for the x-y-plane.

(c) Explain why $f(\hat{r})$ is also a unit vector.

(d) A ray came in the direction $\hat{r}$, hit a surface at a point x at which the normal vector is $\vec{n}$, and was reflected to direction $f(\hat{r})$. To which direction will be reflected a ray that comes in direction $-f(\hat{r})$ (i.e., "comes back") and hits the same surface at the same point x?

(e) Let P be the surface which is the graph of the function $f(x, y) = x^2 + y^2$; this kind of surface is called a *paraboloid.* Calculate the tangent plane to this surface at a given point $(x_0, y_0, f(x_0, y_0))$, and calculate a normal $\vec{n}$ to this plane.

(f) A ray of light coming "from above" (i.e., $\hat{r} = (0, 0, -1)$) hits P at some point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$. In what direction will it be reflected?

(g) Show that regardless of where the ray of light hits P, the reflected ray will pass through the point $(0, 0, 1/4)$, called the *focal point* of the paraboloid.

(h) A ray leaves $(0, 0, 1/4)$ in such a direction that it hits P at a point (x_0, y_0, z_0). In which direction will it be reflected? (Hint: use 3d, 3f, and 3g).

(i) Explain why the facts shown in 3g and 3h make paraboloids useful for making both receiving equipment (e.g., telescopes, satellite dishes, solar cookers) as well as radiating equipment (e.g., spotlights).