18.022 2014, Homework 4. Due Thursday, October 9TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth five points. You can use anything stated in any question, including those you did not solve. Please post a question to the Piazza page if anything is unclear.

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}.$$

  (a) Show that f is differentiable at any $x \neq 0$ by using the fact that a product of differentiable functions is differentiable, as is their composition. You may assume that the relevant standard functions are differentiable. Calculate f's derivative there.
  (b) Show that f is differentiable at $x = 0$ and that the derivative there is equal to zero, by directly using the definition of the derivative.
  (c) Show that the derivative of f is not continuous at $x = 0$, by finding a sequence $x_1, x_2, \ldots$ with $\lim_n x_n = 0$ but $\lim_n \frac{df}{dx}(x_n) \neq 0$.

(2) Let $f_1\colon \mathbb{R}^n \to \mathbb{R}^m$ be given by $f(\vec{v}) = \vec{w} + A \cdot \vec{v}$ for some m-by-n matrix $A = (a_{ij})$ and some vector $\vec{w} \in \mathbb{R}^m$. Show that f is differentiable and calculate its derivative. (Hint: it may help to use Proposition 6 from lecture 9 and Theorem 1 from lecture 10.)

(3) Let $S \subset \mathbb{R}^3$ be given by

$$S = \{(x, y, z) \,|\, x > 0, y > 0, z > 0\}.$$

Let $f\colon S \to \mathbb{R}^3$ be given by

$$f(x, y, z) = (\sqrt{x^2 + y^2}, \tan^{-1}(y/x), z) = (r, \theta, z)$$

the cylindrical coordinates of (x, y, z).

Let $Q \subset \mathbb{R}^2$ be given by

$$Q = \{(x, y) \,:\, x > 0, y > x\}.$$

The *hyperbolic coordinate system* on Q is given by the two coordinates (r, θ), where

$$r = \sqrt{y^2 - x^2} \quad \text{and} \quad \theta = \tanh^{-1}(y/x).$$

Let $g\colon Q \to \mathbb{R}^2$ be given by $g(x, y) = (r, \theta)$.

  (a) Calculate the partial derivatives of f and use Proposition 6 of lecture 9 and Theorem 1 from lecture 10 to explain why f is differentiable and what its derivative Df is.
  (b) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^3$ be the rows of Df at some point $\vec{v}$. What is the relation between these vectors and the three vectors of the orthonormal coordinate system $\hat{e}_r, \hat{e}_\theta$ and $\hat{e}_z$?
  (c) Why is the hyperbolic coordinate system called that?

Omer Tamuz. Email: tamuz@mit.edu.

(d) Show that (unlike the polar, cylindrical and spherical coordinate systems) the hyperbolic coordinate system is *not* an orthogonal coordinate system. That is, the two vectors $(dg_1/dx_1, dg_1/dx_2)$ and $(dg_2/dx_1, dg_2/dx_2)$ are *not* everywhere orthogonal.

(4) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\vec{w} \in \mathbb{R}^n$. Let $\vec{v} \in \mathbb{R}^n$, and let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = f(\vec{w} + x \cdot \vec{v})$. Show that

$$\frac{dg}{dx}(0) = \nabla f(\vec{w}) \cdot \vec{v}.$$

(Hint: use the chain rule.)

(5) An exam hall is a perfect 30-by-30-by-30 meter cube. The differentiable function $T\colon [0, 30]^3 \to \mathbb{R}$ measures the temperature at each point in the hall. Frieda the fruit fly is located at some point $p \in [0, 30]^3$. Frieda would like to move to a warmer point in the hall, but she can only sense the temperature in a very small neighborhood of p. She therefore decides to calculate, for each unit vector $\hat{n}$, by how much the temperature will change if she moved a little in the direction of $\hat{n}$. That is, she wants to calculate

$$d_p(\hat{n}) = \lim_{h\to 0} \frac{T(p + h\hat{n}) - T(p)}{h},$$

since moving in the direction $\hat{n}$ a short distance h will result in a change of about $h \cdot d_p(\hat{n})$ in the temperature.

(a) Write $d_p(\hat{n})$ in terms of $\hat{n}$ and of the gradient (i.e., derivative) of T at p, $\nabla T(p)$. (Hint: note that $d_p(\hat{n})$ is the derivative at $\vec{0}$ of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = T(p + x\hat{n})$, and use question 4.)

(b) Frieda wants to go to where it is warmer, so she decides to move in the direction $\hat{n}$ for which $d_p(\hat{n})$ is the largest. By about how much will the temperature increase if she moves a small distance in that direction?

(c) Frieda has now reached a point $\vec{w}$ in the hall at which she likes the temperature. However, she wants to keep flying around. In which direction(s) can she move in order to go to a place with (about) the same temperature?

(6) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be everywhere differentiable. We say that $p \in \mathbb{R}^n$ is a *maximum point of* f if there exists an $\epsilon > 0$ such that $f(q) < f(p)$ for any $q \in B_\epsilon(p)$ that is not equal to p.

Show that if $p \in \mathbb{R}^n$ is a maximum point then $\nabla f(p) = \vec{0}$, by following these steps:

(a) Deduce from the fact that p is a maximum point that there is an ϵ such that $f(q) \le f(p)$ for all q in $B_\epsilon(p)$.

(b) Assume the contrary - namely that $\nabla f(p) = \vec{w} \ne \vec{0}$. Explain why that means that $\frac{df}{dx_i}(p) \ne 0$ for some i.

(c) Let $\vec{v} = w_i \hat{e}_i$, where $w_i \ne 0$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = f(p + x\vec{v})$. Calculate the derivative of g at 0, using question 4.

(d) Use question 9 to show that there is a $\delta > 0$ such that $g(x) > g(0)$ for all $x \in (0, \delta)$.

(e) Arrive at a contradiction by showing that it is impossible that $f(q) \le f(p)$ for all $q \in B_\epsilon(p)$.

(7) Calculate the tangent hyperplane to the graph of $f\colon \mathbb{R}^n \to \mathbb{R}$ at a maximum point p, using question 6.

(8) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^2$. The *Hessian Matrix* $H = (h_{ij})$ is the n-by-n matrix given by

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

(a) Explain why the Hessian is the derivative of ∇f.

(b) Show that if $h_{ij} = 0$ everywhere then $f(\vec{v}) = a + \vec{w} \cdot \vec{v}$ for some $a \in \mathbb{R}$ and $\vec{w} \in \mathbb{R}^n$. You can use the following theorem: if ∇f is a constant then $f(\vec{v}) = a + \nabla f \cdot \vec{v}$ for some $a \in \mathbb{R}$.

(9) **Bonus question.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable at $x \in \mathbb{R}$, with derivative $\lambda > 0$. Explain why there exists a $\delta > 0$ such that for all positive $h < \delta$ it holds that $f(x + h) > f(x)$.