18.022 2014, Homework 2. Due Thursday, September 25^{TH}

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth three points, except the bonus question, which is worth eight. Four additional points will be given to any assignment in which there is an honest attempt to answer every question (except perhaps the bonus question).

(1) There are n stocks available on the stock market, with prices today given by $\vec{v} = (v_1, v_2, \ldots, v_n)$. Bernie M. buys b_i of stock i, so that his "purchase order vector" is $\vec{b} = (b_1, b_2, \ldots, b_n)$. His "portfolio vector" is $\vec{p} = (v_1 b_1, v_2 b_2, \ldots, v_n b_n)$. Note that each b_i can be negative - this corresponds to buying a negative number of stocks, also called "shorting the stock".

(a) How much did Bernie M. have to pay for his stocks?

(b) Bernie's wife Ruth tries to predict the prices tomorrow. She estimates them to be $\vec{u} = (u_1, u_2, \ldots, u_n)$. What is the amount of money they will they make if her predictions are correct and they sell their stocks tomorrow?

(c) Ruth's predictions are not certain. A measure of the risk of a portfolio $\vec{p}$ is $\sigma = |\vec{p}|$. Assuming $\sum_{i=1}^{n} p_i = \$1,000,000,000$ (that is, the total amount of money invested is one billion), which portfolio minimizes the risk σ? (Hint: write $\vec{p}$ as a sum of its projection $\text{proj}_{\vec{1}}\vec{p}$ and the difference $\vec{p} - \text{proj}_{\vec{1}}\vec{p}$, where $\vec{1} = (1, 1, \ldots, 1)$) What is its risk? Give an example of a portfolio with $\sigma = \$1,000,000,000$.

(2) A gram of vitamin powder brand X has 10% of the recommended daily intake (RDI) of vitamin C, and 40% of the RDI of vitamin D. It costs 50c. A gram of brand Y has 30% and 20% of the RDIs of vitamins C and D, and costs \$1. To summarize, we will write these numbers in a matrix R (of the RDIs) and a vector p (of the prices):

$$R = \begin{pmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{pmatrix} \text{ and } \vec{p} = (\$0.5, \$1)$$

Each column of R corresponds to a brand, and each row corresponds to a vitamin.

Walter mixes v_1 grams of brand X and v_2 grams of brand Y to sell in small plastic bags labeled Z.

(a) What does it cost Walter to produce a bag of Z? Express this also in terms of $\vec{p}$ and $\vec{v}$.

(b) What are the percentages of the RDIs of a bag of Z? Express this also in terms of R and $\vec{v}$.

(c) What is the set of values of $\vec{v}$ that guarantee that a bag of Z contains at least 100% of the RDI of vitamin C? Explain and draw a diagram.

(d) **Bonus question.** Jesse knows a person who is willing to pay \$5 for a bag of Z, as long as it contains at least 100% of the RDI of both vitamins (since that is what it costs to buy 5 grams of Y, which also contain the full RDI). Which values of v_1 and v_2 should Walter choose

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in order to maximize their profits? How much will they make on each bag?

(3) Let $(2s, -2+s, 20)$ and $(4+2t, t, 2t)$ be lines in $\mathbb{R}^3$.
  (a) Show that the two lines intersect. Where is the point of intersection?
  (b) What is the distance between the plane that these lines lie on to the point $(1, 2, 3)$?

(4) Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Let Π be the plane that $\vec{u}$ and $\vec{v}$ lie on (i.e., the one passing through the origin and through their tips), and let Θ be the plane that $\vec{w}$ and $\vec{u}$ lie on.
  (a) Where do Π and Θ intersect?
  (b) What is the distance between Π and the tip of $\vec{w}$?
  (c) What is the (acute or right) angle between Π and Θ?

(5) Let $\vec{u}, \vec{v} \in \mathbb{R}^3$.
  (a) Let $\vec{w} = \lambda\vec{v}$ be parallel to $\vec{v}$. What is $|\vec{w} - \vec{u}|^2$, as a function of λ? Is this a linear function? Quadratic? Other?
  (b) Which value of λ minimizes $|\vec{w} - \vec{u}|$?
  (c) Give a nice expression for $\vec{w}$ when λ is that minimizing value, as well as a geometrical explanation of what is going on.

(6) Let $\theta, \phi \in \mathbb{R}$ be angles, and let M_θ, M_ϕ be the 3×3 matrices given by

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$M_\phi = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

  (a) Give a geometrical description of the linear transformations that these matrices represent.
  (b) Prove that $M_\theta M_\phi \neq M_\phi M_\theta$ by finding a vector $\vec{v} \in \mathbb{R}^3$ and $\theta, \phi \in \mathbb{R}$ such that $M_\theta M_\phi \vec{v} \neq M_\phi M_\theta \vec{v}$.
  (c) Let $\vec{v} = (v_1, v_2, 0)$ and $\vec{w} = (w_1, w_2, 0)$. Explain why

$$(M_\theta \vec{v}) \times (M_\theta \vec{w}) = M_\theta(\vec{v} \times \vec{w}).$$

  Hints: what is the direction of the vector on the left? What is the direction of the vector on the right? What are their lengths?

(7) Let $A = (a_{ij})$ be a 2×2 matrix.
  (a) Let U be the "unit triangle"; its vertices are $\vec{u} = (0, 0)$, $\vec{v} = (1, 0)$, $\vec{w} = (0, 1)$. Let U' be the triangle whose vertices are $A\vec{u}$, $A\vec{v}$ and $A\vec{w}$. What is the area of U'?
  (b) Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ be any vectors, let T be the triangle with vertices at $\vec{u}$ $\vec{v}$ and $\vec{w}$, and let T' be the triangle whose end points are $A\vec{u}$, $A\vec{v}$ and $A\vec{w}$. What is the ratio between the area of T' and the area of T?
  (c) Let $B = (b_{ij})$ also be a 2×2 matrix. For the case that the determinants are positive, explain why the determinant of AB is equal to the determinant of BA which is equal to the product of the determinants of A and B (Note that this is true even if they are not positive).

(8) Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be non-zero, non-parallel vectors. For each of the following functions $f(\vec{w})$, either show that it is linear by showing that $f(\lambda\vec{w}) = \lambda f(\vec{w})$ and $f(\vec{v}+\vec{w}) = f(\vec{v})+f(\vec{w})$, or explain why it is not a linear transformation.

(a) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = \vec{w}$.

(b) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = \vec{0}$.

(c) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = \hat{\imath}$.

(d) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(\vec{w}) = (w_1, w_2)$.

(e) The function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(\vec{w}) = |\vec{w}|$.

(f) The function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(\vec{w}) = \vec{a} \cdot \vec{w}$.

(g) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = \vec{a} + \vec{w}$.

(h) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = \vec{a} \times \vec{w}$.

(i) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = (\vec{a} \times \vec{w}) \cdot \vec{b}$.

(j) The function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{w}) = \mathrm{proj}_{\vec{a}}\vec{w}$.

(k) The function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that reflects a vector around the x-axis.