18.022 2014, Homework 11. Due Monday, December 1st

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth ten points, except the bonus question which is worth five.

(1) Let $\vec{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\vec{F}(x,y) = (0,x)$, and let $\vec{G}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\vec{G}(x,y) = (-y,0)$.

   (a) Use Green's Theorem to show that if $\alpha+\beta \neq 0$ then the area enclosed by any curve C is given by
   $$\frac{1}{\alpha+\beta}\oint_C \left(\alpha\vec{F}+\beta\vec{G}\right)\cdot \mathrm{d}\vec{s}$$

   (b) Use Green's Theorem to calculate the area enclosed between the x-axis and the cycloid $\vec{r}(t) = (t-\sin t, 1-\cos t)$, for $t \in [0,2\pi]$.
   Hint: use $\alpha = 1$ and $\beta = 0$.

   (c) Use Green's Theorem to calculate the area enclosed between the line $\vec{q}(t) = (t, 2t/\pi)$ for $t \in [0,\pi]$ and the cycloid $\vec{r}(t) = (t-\sin t, 1-\cos t)$, again for $t \in [0,\pi]$.
   Hint: use $\alpha = \beta = 1/2$.

   (d) **Bonus question.** Let D be a bounded region of $\mathbb{R}^2$. Recall that the x-coordinate of its center of mass is
   $$\frac{1}{\mathrm{area}(D)}\iint_D x\,\mathrm{d}x\mathrm{d}y$$
   and the y-coordinate of its center of mass is
   $$\frac{1}{\mathrm{area}(D)}\iint_D y\,\mathrm{d}x\mathrm{d}y$$
   Use Green's Theorem to Calculate the center of mass of the region described in part 1b above. Hint: replace $\vec{F}$ and $\vec{G}$ by other appropriate functions.

(2) Let $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a force field. Let $\vec{r}\colon \mathbb{R} \to \mathbb{R}^3$ be a parametrized $\mathcal{C}^1$ curve with $\vec{r}(0) = (a_1,a_2,a_3) = \vec{a}$ and $\vec{r}(1) = (b_1,b_2,b_3) = \vec{b}$. The work done by $\vec{F}$ on a particle moving along $\vec{r}$ from time $t=0$ to time $t=1$ is given by
$$\oint_{\vec{r}([0,1])} \vec{F}\cdot \mathrm{d}\vec{s}.$$
Calculate this work for each of the fields given below.

   (a) $\vec{F}(x,y,z) = (F_1,F_2,F_3)$, for some three constants $F_1,F_2,F_3 \in \mathbb{R}$. I.e., $\vec{F}$ is constant. Express your answer in terms of $F_1,F_2,F_3,\vec{a}$ and $\vec{b}$.

   (b) $\vec{F}(x,y,z) = (x,y,z)$. Express your answer in terms of $\vec{a}$ and $\vec{b}$.

   (c) Let $V\colon \mathbb{R}^3 \to \mathbb{R}$ be $\mathcal{C}^1$, and let $\vec{F}(x,y,z) = \nabla V(x,y,z)$. Express your answer in terms of V, $\vec{a}$ and $\vec{b}$. Hint: Analyze the function $U(t) = V(\vec{r}(t))$.

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(3) Recall that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

In this problem we will prove the following statement. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ harmonic function. Let $C_{x,y,a}$ be the circle of radius a around the point $(x, y) \in \mathbb{R}^2$. Then

$$f(x, y) = \frac{1}{2\pi a} \oint_{C_{x,y,a}} f \, \mathrm{d}s.$$

That is, $f(x, y)$ is equal to the average value of f, taken at a circle of any radius around (x, y). This is called the *mean value theorem* for harmonic functions.

(a) Fix $(x, y) \in \mathbb{R}^2$. Let $g\colon (0, \infty) \to \mathbb{R}$ be given by

$$g(a) = \frac{1}{2\pi a} \oint_{C_{x,y,a}} f \, \mathrm{d}s.$$

Let

$$\vec{F}(x, y) = \left(-\frac{\partial f}{\partial y}(x, y), \frac{\partial f}{\partial x}(x, y) \right).$$

Show that

$$\frac{\mathrm{d}g}{\mathrm{d}a} = \frac{1}{2\pi a} \oint_{C_{x,y,a}} \vec{F} \cdot \mathrm{d}s,$$

where $C_{x,y,a}$ is traversed anti-clockwise. Hint: parametrize $C_{x,y,a}$ by $\vec{r}(\theta) = (a \cos\theta, a \sin\theta)$, with $\theta \in [0, 2\pi]$.

(b) Use Green's Theorem to show that $g' = 0$.

(c) Using the definition of continuity and the fact that f is continuous, explain why $\lim_{a \to 0} g(a) = f(x, y)$. Explain why the mean value theorem for harmonic functions follows from this.
Hint: use the fact that if the values of a function are bounded between some a and b on some domain then its average on that domain will also be between a and b.

(d) Use the mean value theorem for harmonic functions to prove the *maximum principle for harmonic functions*: f cannot have a strict local maximum (i.e., a point $(x, y) \in \mathbb{R}^2$ such that $f(x, y) > f(x', y')$ for all $(x', y') \neq (x, y)$ in some small enough ball around (x, y)).