18.022 2014, Homework 10. Due Thursday, November 20th

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth eight points, except the bonus question, which is worth five. Four additional points will be given to any assignment in which there is an honest attempt to answer every question (except perhaps the bonus question).

(1) Recall that the *gravitational force* exerted by an object at $\vec{r}$ of mass M on an object at $\vec{r}_0$ of mass m is

$$\vec{F} = -\frac{GmM}{|\vec{r}_0 - \vec{r}|^3}(\vec{r}_0 - \vec{r}).$$

The *gravitational potential* is

$$U = -\frac{GmM}{|\vec{r}_0 - \vec{r}|}.$$

or, if we denote $\vec{r}_0 = (x_0, y_0, z_0)$ and $\vec{r} = (x, y, z)$, then equivalently

$$U(x_0, y_0, z_0) = -\frac{GmM}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}.$$

It is easy to verify that $\vec{F} = -\nabla U$. Note that unfortunately the the definition of "potential" in physics differs from the mathematical one by a minus sign. In this question we follow the physics convention.

Consider an object that occupies a solid region $W \subset \mathbb{R}^3$, and has density δ, where the density is defined as the ratio of mass to to volume. The gravitational potential induced by this object at $\vec{r}$ is

$$U(x_0, y_0, z_0) = -\iiint_W \frac{Gm\delta}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

The idea is that potential is additive; the potential induced by a number of objects is the sum of the potential. Hence the potential induced by an object occupying a region can be calculated by diving the region into infinitesimal boxes and summing (i.e., integrating) the potential from each box. The mass of an an infinitesimal box with volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ is $\delta\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Fix $0 < R_1 < R_2$. Let $W = \{(x, y, z) \in \mathbb{R}^3 : R_1^2 \leq x^2 + y^2 + z^2 \leq R_2^2\}$ be the set of points which are in the sphere of radius R_2 but not in the sphere of radius R_1.

(a) Let δ be such that the mass of the object is M. What is δ?

(b) Let $(x_0, y_0, z_0) = (0, 0, a)$ with $a > R_2$, so that (x_0, y_0, z_0) is outside the sphere of radius R_2. Calculate $U(x_0, y_0, z_0)$ in terms of G, M, m and a, and show that it does not depend explicitly on the radii R_1 and R_2. How does this compare to the potential induced at $(0, 0, a)$ by a point mass M concentrated at the origin? (Hint: use spherical coordinates.)

(c) Let $(x_0, y_0, z_0) = (0, 0, a)$ for some $0 < a < R_1$. What is $U(x_0, y_0, z_0)$? What is the force $\vec{F}$ exerted at this point (recall that $\vec{F} = -\nabla U$)? (Hint: the force in the x and y directions is zero, by symmetry.)

(2) In this question we will learn a little about what the *Fourier transform* is.

Let $\mathcal{F}$ be the family of functions $f \colon [-1, 1] \to \mathbb{R}$ with the following properties:

- f is continuous.
- f is even (i.e., $f(-x) = f(x)$ for all $x \in [-1, 1]$).

Omer Tamuz. Email: tamuz@mit.edu.

Given $f, g \in \mathcal{F}$, define the "dot product" between f and g by

$$(f, g) = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x,$$

and accordingly define the norm of f by

$$|f| = \sqrt{(f, f)}.$$

or

$$|f|^2 = \int_{-1}^{1} f(x)^2\,\mathrm{d}x.$$

For $n > 0$, let $c_n \colon [-1, 1] \to \mathbb{R}$ be given by

$$c_n(x) = \cos(n\pi x),$$

and for $n = 0$ let $c_0 \colon [-1, 1] \to \mathbb{R}$ be the constant function $c_0(x) = 1/\sqrt{2}$. Note that each c_n is in $\mathcal{F}$.

(a) Explain why every $f \in \mathcal{F}$ is bounded, and that therefore its norm is finite.
(b) Let f, g be in $\mathcal{F}$, and let $\lambda \in \mathbb{R}$. Verify that $f+g \in \mathcal{F}$ and that $\lambda f \in \mathcal{F}$.
(c) Calculate the norm of each c_n.
(d) Calculate the dot product (c_n, c_k), given n and k. Hint: use the identity

$$2\cos(n\pi x)\cos(k\pi x) = \cos((n-k)\pi x) + \cos((n+k)\pi x).$$

(e) Let $g(x) = a_1c_1(x) + a_2c_2(x) + \cdots + a_mc_m(x)$ for some vector of constants $\vec{a} = (a_1, \ldots, a_m)$. Show that $a_k = (g, c_k)$.
(f) Let g be as in the previous question, and let $f(x) = b_1c_1(x)+b_2c_2(x)+\cdots + b_mc_m(x)$ for some constant $\vec{b} = (b_1, \ldots, b_m)$. Show that the dot product (f, g) is equal to the dot product $\vec{a} \cdot \vec{b}$, and that norm of g is equal to the norm of $\vec{a}$. (Hint: try it first for small values of m (say 2 and 3) to understand why it is true in general.)
(g) **Bonus question.** As you will learn in more advanced courses, every $f \in \mathcal{F}$ can be written as the (perhaps infinite) sum $f(x) = \sum_{n=0}^{\infty} a_nc_n(x)$, where $a_n = (f, c_n)$. Use this to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

(Hint: consider $f(x) = x^2$.)

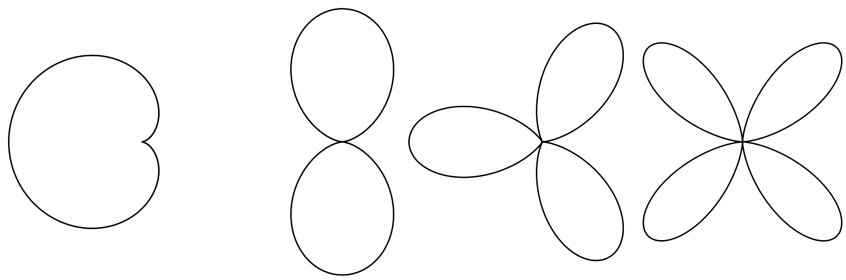

FIGURE 1. The curves $r = 1 - \cos(n\theta)$ for $n = 1, 2, 3, 4$.

(3) Recall that we showed in class that the cardioid, the region enclosed by the curve $r = 1 - \cos\theta$, has area $3\pi/2$. We are interested in the region enclosed by the curve $r = 1 - \cos(2\theta)$. More generally, let D be the region enclosed by the curve $r = |g(\theta)|$, for some continuous $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x + 2\pi) = g(x)$ for all $x \in \mathbb{R}$. Given $n \in \mathbb{N}$, we would like to find the area of the region D_n enclosed by the curve $r = g(n\theta)$.

(a) Calculate the area of D in terms of a one-dimensional integral involving g.

(b) Calculate the area of D_n, in terms of the area of D and of n. What is the area enclosed by $r = 1 - \cos(2\theta)$?

(c) Using question 2, calculate the area enclosed by $r = |a + b\cos(\theta) + c\cos(2\theta) + d\cos(3\theta)|$, for some constants a, b, c, d. (Hint: use the change of variables $\pi x + \pi = \theta$.)