18.022 2014, Homework 1
Due Thursday, September 18th

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each problem is worth five points.

(1) Let $\vec{v} = (1, 2)$.
  (a) Write in parametric form the set of vectors in $\mathbb{R}^2$ that are parallel to $\vec{v}$.
  (b) Which of these have magnitude 2?

(2) Two roads diverged in a yellow wood, located in $O = (6, -8)$. One went to $P = (8, -7)$ and the other to $R = (5, -6)$. Let $\vec{v} = \overrightarrow{OP}$ and $\vec{w} = \overrightarrow{OR}$.
  (a) Calculate $\vec{v}$ and $\vec{w}$.
  (b) Calculate the magnitudes of $\vec{v}$ and $\vec{w}$.
  (c) Calculate $\vec{v} - \vec{w}$.
  (d) Use the answer to the previous question to calculate the distance between P and R.

(3) Let $\vec{v} = (-4, 3, 2)$, $\vec{u} = (4, 5, 6)$ and $\vec{w} = (8, 8, x)$, for some $x \in \mathbb{R}$. Consider the two lines given by $\vec{v} + t\vec{u}$ and $2\vec{v} + s\vec{w}$.
  (a) For which values of x do these lines intersect?
  (b) When they do intersect, where is the point of intersection?
  (c) For which values of y are the lines $(1, 2, 3) + t(4, 5, y)$ and $(5, 7, 9) + s(-8, -10, -12)$ the same line?

(4) Let $\vec{v} \cdot \vec{w} = 0$.
  (a) Write the magnitude of $\vec{v} + \vec{w}$ in terms of the magnitudes of $\vec{v}$ and $\vec{w}$.
  (b) Write the magnitude of $\vec{v} - \vec{w}$ using the magnitudes of $\vec{v}$ and $\vec{w}$.

(5) Let $\vec{v} = (\cos\theta, \sin\theta)$ and $\vec{w} = (\cos\alpha, \sin\alpha)$, for some angles θ and α.
  (a) Calculate $\vec{v} \cdot \vec{w}$. Can you give the result a geometric interpretation?
  (b) Calculate $|\vec{v} - \mathrm{proj}_{\vec{w}}\vec{v}|$. Can you give the result a geometric interpretation?

(6) Let $\theta = \pi/4$, let $\vec{v} = (\cos\theta, \sin\theta)$ and let $\vec{u} = (-\sin\theta, \cos\theta)$.
  (a) Write $\vec{w} = (\sqrt{2}, \sqrt{8})$ as a sum of the form $\lambda_1\vec{v} + \lambda_2\vec{u}$, where λ_1 and λ_2 are scalars.
  (b) Show how any vector $\vec{w} = (w_1, w_2)$ can be written as such a sum.
  (c) Let α be an angle, and let $\vec{v}_1 = (\cos\alpha, \sin\alpha)$ and $\vec{v}_2 = (-\sin\alpha, \cos\alpha)$. Show how any vector $\vec{w} = (w_1, w_2)$ can be written as a sum of the form $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2$. Hint: prove that $\lambda_1 = \vec{v}_1 \cdot \vec{w}$ and $\lambda_2 = \vec{v}_2 \cdot \vec{w}$ work.
  (d) Let $\vec{v}_1$ and $\vec{v}_2$ be as in the previous problem. Explain why any vector $\vec{w}$ is equal to $\mathrm{proj}_{\vec{v}_1}\vec{w} + \mathrm{proj}_{\vec{v}_2}\vec{w}$.
  (e) Let $\vec{v}, \vec{u}$ be vectors in $\mathbb{R}^3$. Explain why there exist vectors in $\mathbb{R}^3$ that cannot be written as $\lambda_1\vec{v} + \lambda_2\vec{u}$. (Hint: what is the shape of the set of vectors that can be written like that?)

(7) Let $\vec{v} = (1, 2, 3)$.

Omer Tamuz. Email: tamuz@mit.edu.

(a) Show that $\vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$ for any $\vec{u}$. (Hint: the fact that $\vec{v} = (1, 2, 3)$ doesn't matter; this holds for any non-zero vector $\vec{v}$.)
(b) Find a vector $\vec{w}$ that is orthogonal to $\vec{v}$ and has magnitude 1.
(c) **Bonus question.** Find a vector $\vec{u}$ that is orthogonal both to $\vec{v}$ and to $\vec{w}$, and has magnitude 1.