Blackboard 1. *A **function** $f\colon A \to B$ consists of:*

(1) *the set A, which is called the **domain** of f,*
(2) *the set B, which is called the **range** of f,*
(3) *a rule which assigns to every element $a \in A$ an element $b \in B$, which we denote by $f(a)$.*

It is easy to write down examples of functions:

(1) Let A be the set of all people and let $B = [0, \infty)$. Let $f(x)$ be the height of person x, to the nearest inch.
(2) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
(3) $f\colon \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$.
(4) $f\colon [0, \infty) \to \mathbb{R}$ given by $f(x) = x^2$.
(5) $f\colon [0, \infty) \to [0, \infty)$ given by $f(x) = x^2$.
(6) The Dirichlet function $D\colon \mathbb{R} \to \{0, 1\}$ given by

$$D(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

(7) The parity function $p\colon \mathbb{N} \to \{0, 1\}$ given by

$$p(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd.} \end{cases}$$

Blackboard 2. *If $f\colon A \to B$ and $g\colon B \to C$ are two functions, then we can **compose** f and g to get a new function $g \circ f\colon A \to C$. If $a \in A$, then $(g \circ f)(a) = g(f(a))$.*

If $f\colon \mathbb{R} \to \mathbb{R}$ is the function $f(x) = \sin x$ and $g\colon \mathbb{R} \to \mathbb{R}$ is the function $g(x) = 2x$, then $g \circ f\colon \mathbb{R} \to \mathbb{R}$ is the function $(g \circ f)(x) = 2\sin x$, whilst $f \circ g\colon \mathbb{R} \to \mathbb{R}$ is the function $(f \circ g)(x) = \sin 2x$.

If $f\colon$ "people" $\to$ "sequences of words" assigns to a person her / his name and $g\colon$ "sequences of words" $\to$ "words" chooses the first word in the sequence, then $g \circ f\colon$ "people" $\to$ "words" assigns to each person her / his first name.

Blackboard 3. *Let $f\colon A \to B$ be a function.*
*We say that f is **injective** if whenever $f(a_1) = f(a_2)$ for some pair a_1, $a_2 \in A$, then in fact $a_1 = a_2$. Also called **one-to-one**.*
*We say that f is **surjective** if for every $b \in B$, we may find an $a \in A$ such that $f(a) = b$. Also called **onto**.*
*We say that f is a **bijection** if f is both injective and surjective.*

It is interesting to go through the examples above. The function in (1) is neither injective or surjective. Indeed there are lots of people who are five foot nine, so just because there are two people with the same height, does not mean they are the same person. So f is not injective. On the other hand, there is no person who is a million inches high, so f is not surjective either.

The function in (2) is neither injective nor surjective as well. $f(-1) = 1 = f(1)$, but $1 \neq -1$. There is no real number whose square is -1, so there is no real number a such that $f(a) = -1$.

The function in (3) is not injective but it is surjective. $f(-1) = f(1)$, and $1 \neq -1$. But if $b \geq 0$ then there is always a real number $a \geq 0$ such that $f(a) = b$ (namely, the square root of b).

The function in (4) is injective but not surjective. If $f(a_1) = f(a_2)$, then $a_1^2 = a_2^2$. As both $a_1 \geq 0$ and $a_2 \geq 0$, this implies $a_1 = a_2$. On the other hand, there is still no number whose square is -1.

The function in (5) is bijective. It is injective, as in (4) and it is surjective as in (3).

The function in (6) is not injective but it is surjective. $f(0) = 0 = f(1)$, but $0 \neq 1$. On the other hand, if $b \in \{0, 1\}$ then either $b = 0$ or $b = 1$. But $f(0) = 0$ and $f(\sqrt{2}) = 1$.

The function in (7) is not injective but it is surjective. $f(0) = f(2) = 0$, but $0 \neq 2$. On the other hand, if $b \in \{0, 1\}$ then either $b = 0$ or $b = 1$. But $f(0) = 0$ and $f(1) = 1$.

(8) $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(\vec{v}) = \|v\|$.
(9) $f: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ given by $f(\vec{v}) = \frac{\vec{v}}{\|v\|}$.
(10) $r: \mathbb{R} \to \mathbb{R}^3$ given by $f(t) = (\cos t, \sin t, t)$.
(11) $f: \mathbb{R} \to (0, \infty)$ is given by $f(x) = e^x$.

The function in (8) is neither injective nor surjective. There are plenty of unit vectors and there are no vectors of negative length.

The function in (9) is neither injective nor surjective. There are plenty of vectors which point in the same direction and the image consists of vectors of unit length.

The function in (10) is injective but not surjective.

The function in (11) is bijective. If $f: A \to B$ is a bijective function, then f has an inverse function $g: B \to A$. $g \circ f: A \to A$ is the identity - it sends a to a. Similarly $f \circ g: B \to B$ is the identity - it sends b to b. The inverse of the exponential function is the logarithm $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \ln x$.

Blackboard 4. *If A and B are two sets, the **product** of A and B, denoted $A \times B$ is the set of ordered pairs*

$$A \times B = \{ (a, b) \,|\, a \in A, b \in B \}.$$

The product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, $\mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$, and so on.

(12) Let P be the set of odd primes: $P = \{3, 5, 7, 11, \ldots\}$. Let E be the set of even numbers that are greater than 4: $E = \{6, 8, 10, \ldots\}$. Let $g: P \times P \to E$ be given by $g(p_1, p_2) = p_1 + p_2$.

The function in (12) is not injective. For example, $g(5, 7) = g(7, 5) = 12$. Is it surjective? The *Goldbach Conjecture* can be stated as: f is surjective. This is not known.

Blackboard 5. *The **graph** of a function $f: A \to B$ is the subset of the product*

$$\{ (a, f(a) \,|\, a \in A \} \subset A \times B.$$

If $f: \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^2$, then the graph of f is the parabola, $y = x^2$,

$$\{ (x, x^2) \,|\, x \in \mathbb{R} \} \subset \mathbb{R}^2.$$

Now suppose that $g: \mathbb{R}^2 \to \mathbb{R}$ is the function

$$g(x, y) = x^2 + y^2.$$

The graph of g is the set $z = x^2 + y^2$,
$$\{ (x, y, x^2 + y^2) \,|\, (x, y) \in \mathbb{R}^2 \} \subset \mathbb{R}^3.$$

One way to picture g is to slice it using level curves and level sets.

If $g\colon \mathbb{R}^2 \to \mathbb{R}$ is any function, the level curve of g at height c is the set
$$\{ (x, y) \in \mathbb{R}^2 \,|\, g(x, y) = c \} \subset \mathbb{R}^2.$$

In the example when $f(x, y) = x^2 + y^2$, the level curves are concentric circles of radius $\sqrt{c}$, centred at the origin. If $c = 0$ the level curve is the origin and if $c < 0$ the level curve is empty. Note that graph has a minimum at the origin.

If $g\colon \mathbb{R}^3 \to \mathbb{R}$ is any function, the level set of g at height c is the set
$$\{ (x, y, z) \in \mathbb{R}^3 \,|\, g(x, y, z) = c \} \subset \mathbb{R}^3.$$

If $g(x, y, z) = z - x^2 - y^2$ then the level sets are parallel paraboloids.

It is interesting to try to visualise various functions. If we consider $f(x, y) = -x^2 - y^2$, then this is an upside down paraboloid, which has a maximum at the origin. Note that the level curves are the same as the level curves for $f(x, y) = x^2 + y^2$ (for different values of c, of course).

Now consider the function $f(x, y) = x^2 - y^2$. If we consider fixing $y = b$ and varying x, then we get a parabola $z = x^2 - b^2$. If we consider fixing $x = a$ and varying y, then we get an upside down parabola, $z = a^2 - y^2$. In fact the origin is a saddle point, a point which is neither a maximum nor a minimum.