THE DIVERGENCE THEOREM
BASED ON LECTURE NOTES BY JAMES MCKERNAN AND PAVEL ETINGOF

The Divergence Theorem is also known as Gauss's Theorem, and as Ostrogradsky's Theorem. It was first discovered by Lagrange in 1762, and then independently by Gauss in 1813 and by Ostrogradsky in 1826. Ostrogradsky gave the first general proof.

Theorem 1 (Stigler's law of eponymy)**.** *No scientific discovery is named after its original discoverer.*

This was discovered by the sociologist Robert Merton.

Example 2. *Some examples from mathematics:*

- *The Gaussian distribution was discovered by de Moivre.*
- *L'Hôpital's rule was discovered by Johann Bernoulli.*
- *Euler's constant was discovered by Jacob Bernoulli.*
- *The Cantor-Bernstein-Schröder Theorem was discovered by Dedekind.*
- *Burnside's lemma was discovered by Cauchy. Burnside himself attributed it to Frobenius.*

Theorem 3 (The Divergence Theorem)**.** *Let $M \subset \mathbb{R}^3$ be a compact subset whose boundary ∂M is a differentiable parametrized two dimensional surface. Let $\vec{F} \colon M \longrightarrow \mathbb{R}^3$ be a smooth vector field.*

Then

$$\iiint_M \operatorname{div} \vec{F} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\partial M} \vec{F} \cdot \mathrm{d}\vec{S},$$

where ∂M is given the outward orientation.

We will prove this theorem for that case that M is a region of type I, II and III.

Proof. Suppose D is a subset of the x-y-plane, and that

$$M = \{(x, y, z) \, : \, (x, y) \in D, a(x, y) \leq z \leq b(x, y)\}.$$

Suppose $\vec{F} = (0, 0, F_3)$. Then

$$\begin{aligned} \iiint_M \operatorname{div} \vec{F} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \iiint_M \frac{\partial F_3}{\partial z} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \iint_D \int_{a(x,y)}^{b(x,y)} \frac{\partial F_3}{\partial z} \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y \\ &= \iint_D \left(F_3(x, y, b(x, y)) - F_3(x, y, a(x, y)) \right), \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Now, divide the boundary ∂M into the top part, ∂M^+, and the bottom part, ∂M^-. We can parametrize the top part by $\vec{r}(x, y) = (x, y, b(x, y))$. Then

$$\iint_{\partial M^+} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_D \vec{F}(\vec{r}(x, y)) \cdot (\vec{T}_x(x, y) \times \vec{T}_y(x, y)) \, \mathrm{d}x \, \mathrm{d}y.$$

Now,

$$\vec{T}_x = (1, 0, b_x)$$

and

$$\vec{T}_y = (0, 1, b_y),$$

and so

$$\vec{F} \cdot (\vec{T}_x \times \vec{T}_y) = \begin{pmatrix} 0 & 0 & F_3 \\ 1 & 0 & b_x \\ 0 & 1 & b_y \end{pmatrix} = F_3.$$

Hence

$$\iint_{\partial M^+} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_D F_3(x, y, b(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

By a similar argument,

$$\iint_{\partial M^-} \vec{F} \cdot \mathrm{d}\vec{S} = -\iint_D F_3(x, y, a(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Where the minus sign is due to the fact that in ∂M^- we need to orient the normal downwards.

We have thus shown the claim for $\vec{F}$ of the form $\vec{F} = (0, 0, F_3)$. For general $\vec{F}$, split each integral into three:

$$\iiint_M \operatorname{div} \vec{F}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_M \frac{\partial F_1}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_M \frac{\partial F_2}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_M \frac{\partial F_3}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and

$$\iint_{\partial M} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{\partial M} (F_1, 0, 0) \cdot \mathrm{d}\vec{S} + \iint_{\partial M} (0, F_2, 0) \cdot \mathrm{d}\vec{S} + \iint_{\partial M} (0, 0, F_3) \cdot \mathrm{d}\vec{S}.$$

We already showed that the last addends are equal. To show that the first are equal and that the second are equal, apply the same proof, but using the characterization of M as a region of the other types. $\square$

Example 4. *Calculate the flux of $\vec{F}(x, y, z) = (x, 2y, 3z)$ through the ellipsoid M given by $x^2 + y^2/3 + z^2/3 = 1$.*

$\operatorname{div} \vec{F} = 6$. *Hence the answer is $6 \operatorname{vol}(M)$. The volume of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $\frac{4}{3}\pi abc$. Hence*

$$6 \operatorname{vol}(M) = 6\frac{4}{3}\pi \cdot \sqrt{1 \cdot 3 \cdot 3} = 24\pi.$$