MORE ON CONSERVATIVE VECTOR FIELDS
BASED ON LECTURE NOTES BY JAMES MCKERNAN

Definition 1. *A region $D \subseteq \mathbb{R}^2$ is* convex *if for every $p, q \in D$ the line segment $\overline{pq}$ is in D.*

A region $D \subseteq \mathbb{R}^2$ is star shaped *if there exists a point $p \in D$ such that for every $q \in D$ the line segment $\overline{pq}$ is in D.*

Theorem 2. *If $D \subseteq \mathbb{R}^2$ is star shaped and $\vec{F}\colon D \to \mathbb{R}^2$ is a $\mathcal{C}^\infty$ vector field in D then $\vec{F}$ is conservative if and only if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.*

The condition on the partial derivatives can be thought of as saying that $\vec{F}$ has zero curl.

Proof. We already saw that every conservative vector field has zero curl. We also saw that this is not true in general, and so we will need the star shaped property to prove the other direction.

Define $f\colon D \to \mathbb{R}$ by

$$f(q) = \oint_{\overline{pq}} \vec{F} \cdot \mathrm{d}\vec{s},$$

where $\overline{pq}$ is oriented from p to q. Let's compute $\frac{\partial f}{\partial x}$. Let $q' = q + h\hat{e}_1$. Then

$$\begin{aligned}\frac{\partial f}{\partial x}(q) &= \lim_{h \to 0} \frac{f(q') - f(q)}{h} \\ &= \lim_{h \to 0} \frac{\oint_{\overline{pq'}} \vec{F} \cdot \mathrm{d}\vec{s} - \oint_{\overline{pq}} \vec{F} \cdot \mathrm{d}\vec{s}}{h}.\end{aligned}$$

Since $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, it follows from Green's Theorem that

$$\oint_{\overline{pq'}} \vec{F} \cdot \mathrm{d}\vec{s} = \oint_{\overline{pq}} \vec{F} \cdot \mathrm{d}\vec{s} + \oint_{\overline{qq'}} \vec{F} \cdot \mathrm{d}\vec{s}.$$

So we get

$$\frac{\partial f}{\partial x}(q) = \lim_{h \to 0} \frac{\oint_{\overline{qq'}} \vec{F} \cdot \mathrm{d}\vec{s}}{h} = F_1(q).$$

Similarly, $\frac{\partial f}{\partial y}(q) = F_2(q)$. Hence $\vec{F} = \nabla f$ and $\vec{F}$ is conservative. □

Definition 3. *Let $\vec{F}\colon U \longrightarrow \mathbb{R}^2$ be a vector field. Define another vector field by the rule*

$$\vec{F}^*\colon U \longrightarrow \mathbb{R}^2 \qquad \text{where} \qquad \vec{F}^* = (-F_2, F_1).$$

Theorem 4 (Divergence theorem in the plane)**.** *Suppose that $M \subset \mathbb{R}^2$ is a smooth 2-manifold with boundary ∂M.*

If $\vec{F}\colon U \longrightarrow \mathbb{R}^2$ is a smooth vector field, then

$$\iint_M \operatorname{div} \vec{F}\, \mathrm{d}A = \int_{\partial M} \vec{F} \cdot \hat{n}\, \mathrm{d}s,$$

where $\hat{n}$ is the unit normal vector of the smooth oriented curve $C = \partial M$ which points out of M.

Proof. Note that

$$\operatorname{curl}(\vec{F}^*) = \operatorname{div} \vec{F},$$

and

$$\vec{F}^* \cdot \mathrm{d}\vec{s} = (\vec{F} \cdot \hat{n})\mathrm{d}s,$$

and so the result follows from Green's theorem applied to $\vec{F}^*$. □