# Change of coordinates: II
## Based on lecture notes by James McKernan

**Example 1.** *Let $D$ be the region bounded by the cardioid,*

$$r = 1 - \cos\theta.$$

*If we multiply both sides by $r$ and take $r\cos\theta$ over the other side, then we get*

$$(x^2 + y^2 + x)^2 = x^2 + y^2.$$

*We have*

$$\begin{aligned}
\operatorname{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left( \int_0^{1-\cos\theta} r\,\mathrm{d}r \right) \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left[ \frac{r^2}{2} \right]_0^{1-\cos\theta} \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{(1-\cos\theta)^2}{2}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \cos\theta + \frac{\cos^2\theta}{2}\,\mathrm{d}\theta \\
&= \left[ \frac{\theta}{2} - \sin\theta \right]_{-\pi}^{\pi} + \frac{\pi}{2} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

In $\mathbb{R}^3$, we can either use cylindrical or spherical coordinates, instead of Cartesian coordinates.

Let's first do the case of cylindrical coordinates. Recall that

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z.
\end{aligned}$$

So the Jacobian is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

So,

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(r,\theta,z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

**Example 2.** *Consider a ball of radius $a$. Put the centre of the ball at the point $(0,0,0)$. Note that*

$$x^2 + y^2 + z^2 = a^2,$$

*translates to the equation*

$$r^2 + z^2 = a^2,$$

*so that*

$$r = \sqrt{a^2 - z^2}.$$

$$\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint_{W^*} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_{-a}^{a} \left( \int_0^{2\pi} \left( \int_0^{\sqrt{a^2-z^2}} r\,\mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}z \\
&= \frac{1}{2} \int_{-a}^{a} \left( \int_0^{2\pi} \left[ r^2 \right]_0^{\sqrt{a^2-z^2}} \mathrm{d}\theta \right) \mathrm{d}z \\
&= \frac{1}{2} \int_{-a}^{a} \left( \int_0^{2\pi} a^2 - z^2\,\mathrm{d}\theta \right) \mathrm{d}z \\
&= \pi \int_{-a}^{a} a^2 - z^2\,\mathrm{d}z \\
&= \pi \left[ a^2 z - \frac{z^3}{3} \right]_{-a}^{a} \\
&= \frac{4\pi a^3}{3}.
\end{aligned}$$

Now consider using spherical coordinates. Recall that

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\phi.
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}(\rho, \phi, \theta) &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi.
\end{aligned}$$

Notice that this is greater than zero, if $0 < \phi < \pi$. So,

$$\iiint_W f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(\rho, \phi, \theta)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

**Example 3.** *Consider a ball of radius $a$. Put the centre of the ball at the point $(0,0,0)$.*

$$\begin{aligned}
\operatorname{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_{W^*} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left( \int_0^{\pi} \left( \int_0^{a} \rho^2 \sin\phi\,\mathrm{d}\rho \right) \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left( \int_0^{\pi} \sin\phi \left[ \frac{\rho^3}{3} \right]_0^a \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left( \int_0^{\pi} \sin\phi \frac{a^3}{3}\,\mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} [-\cos\phi]_0^{\pi}\,\mathrm{d}\theta \\
&= \frac{2a^3}{3} \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{4\pi a^3}{3}.
\end{aligned}$$