Change of coordinates: I
Based on lecture notes by James McKernan

Blackboard 1. *A function $f\colon U \longrightarrow V$ between two open subsets of $\mathbb{R}^n$ is called a* ***diffeomorphism*** *if:*

(1) *f is a bijection,*
(2) *f is differentiable, and*
(3) *f^{-1} is differentiable.*

By definition of the inverse function, $f \circ f^{-1}\colon V \longrightarrow V$ and $f^{-1} \circ f\colon U \longrightarrow U$ are both the identity function, so that

$$(f \circ f^{-1})(\vec{y}) = \vec{y} \qquad \text{and} \qquad (f^{-1} \circ f)(\vec{x}) = \vec{x}.$$

It follows that

$$Df(\vec{x})Df^{-1}(\vec{y}) = I_n \qquad \text{and} \qquad Df^{-1}(\vec{y})Df(\vec{x}) = I_n,$$

by the chain rule. It follows that

$$\det(Df) \neq 0.$$

and

$$Df^{-1} = (Df)^{-1}.$$

Example 2. *Let $g(r, \theta) = (r\cos\theta, r\sin\theta)$. Then*

$$Dg(r, \theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix},$$

so that

$$\det Dg(r, \theta) = r.$$

Theorem 3. *Let $g\colon U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$,*

$$g(u, v) = (x(u, v), y(u, v)).$$

Let $D^* \subset U$ be a region and let $D = f(D^*) \subset V$. Let $f\colon D \longrightarrow \mathbb{R}$ be a function.
Then

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D^*} f(x(u, v), y(u, v))|\det Dg(u, v)|\,\mathrm{d}u\,\mathrm{d}v.$$

It is convenient to use the following notation:

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \det Dg(u, v).$$

The LHS is called the **Jacobian**. Note that

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \left(\frac{\partial(u, v)}{\partial(x, y)}(x, y)\right)^{-1}.$$

Example 4. *Let $g(r, \theta) = (r\cos\theta, r\sin\theta)$. Then*

$$\det Dg(r, \theta) = r,$$

and so

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D^*} f(x(u, v), y(u, v))r\,\mathrm{d}u\,\mathrm{d}v.$$

Example 5. *There is no simple expression for the integral of* e^{-x^2}. *However it is possible to compute the following integral*

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

(In what follows, we will ignore issues relating to the fact that the integrals are improper; in practice all integrals converge). Instead of computing I, *we compute* I^2,

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathbb{R}^2} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\infty}\left(\int_0^{2\pi} re^{-r^2}\,\mathrm{d}\theta\right)\mathrm{d}r \\
&= \int_0^{\infty} re^{-r^2}\left(\int_0^{2\pi}\mathrm{d}\theta\right)\mathrm{d}r \\
&= 2\pi\int_0^{\infty} re^{-r^2}\,\mathrm{d}r \\
&= 2\pi\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} \\
&= \pi.
\end{aligned}$$

So $I = \sqrt{\pi}$.

Example 6. *Find the area of the region* D *bounded by the four curves*

$$xy = 1, \qquad xy = 3, \qquad y = x^3, \qquad \textit{and} \qquad y = 2x^3.$$

Define two new variables,

$$u = \frac{x^3}{y} \qquad \textit{and} \qquad v = xy.$$

Then D *is a rectangle in uv-coordinates,*

$$D^* = [1/2, 1] \times [1, 3]$$

Now for the Jacobian we have

$$\frac{\partial(u,v)}{\partial(x,y)}(x,y) = \begin{vmatrix} \frac{3x^2}{y} & -\frac{x^3}{y^2} \\ y & x \end{vmatrix} = \frac{4x^3}{y} = 4u.$$

It follows that

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \frac{1}{4u}.$$

This is nowhere zero. In fact note that we can solve for x and y explicitly in terms of u and v.

$$uv = x^4 \qquad \textit{and} \qquad y = \frac{x}{v}.$$

So

$$x = (uv)^{1/4} \qquad \textit{and} \qquad y = u^{-1/4}v^{3/4}.$$

Therefore

$$\begin{aligned}
\operatorname{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} \frac{1}{4u}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 \left(\int_{1/2}^1 \frac{1}{u}\,\mathrm{d}u\right)\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 [\ln u]_{1/2}^1\,\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 \ln 2\,\mathrm{d}v \\
&= \frac{1}{2}\ln 2.
\end{aligned}$$

Theorem 7. *Let $g\colon U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^3$,*

$$g(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)).$$

Let $W^* \subset U$ be a region and let $W = f(W^*) \subset V$. Let $f\colon W \longrightarrow \mathbb{R}$ be a function.

Then

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\det Dg(u,v,w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

As before, it is convenient to introduce more notation:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}(u,v,w) = \det Dg(u,v,w).$$