Double integrals
Based on lecture notes by James McKernan

Definition 1. *Let $S = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle in the plane. A **partition** $\mathcal{P}$ of S is a pair of sequences:*

$$a = x_0 < x_1 < \cdots < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_n = d.$$

*The **mesh** of $\mathcal{P}$ is*

$$m(\mathcal{P}) = \max\{\, x_i - x_{i-1}, y_i - y_{i-1} \,|\, 1 \leq i \leq k \,\}.$$

Now suppose we are given a function

$$f\colon S \longrightarrow \mathbb{R}$$

Pick

$$\vec{c}_{ij} \in S_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

Definition 2. *The sum*

$$T_{\mathcal{P}} = \sum_{i=1}^{n} \sum_{j=1}^{n} f(\vec{c}_{ij})(x_i - x_{i-1})(y_j - y_{j-1}),$$

*is called a **Riemann sum**.*

We will use the short hand notation

$$\Delta x_i = x_i - x_{i-1} \qquad \text{and} \qquad \Delta y_j = y_j - y_{j-1}.$$

Definition 3. *The function $f\colon S \longrightarrow \mathbb{R}$ is called **integrable**, with integral I, if for every $\epsilon > 0$, we may find a $\delta > 0$ such that for every mesh $\mathcal{P}$ whose mesh size is less than δ, we have*

$$|I - T_{\mathcal{P}}| < \epsilon,$$

where $T_{\mathcal{P}}$ is any Riemann sum associated to $\mathcal{P}$.

We write

$$\iint_S f(x, y)\, \mathrm{d}x\, \mathrm{d}y = I,$$

to mean that f is integrable with integral I.

We use a sneaky trick to integrate over regions other than rectangles. Suppose that D is a bounded subset of the plane. Then we can find a rectangle S which completely contains D.

Definition 4. *The **indicator function** of $D \subset S$ is the function*

$$\mathbb{1}_D\colon S \longrightarrow \mathbb{R},$$

given by

$$\mathbb{1}_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D. \end{cases}$$

*If $\mathbb{1}_D$ is integrable, then we say that the **area of** D is the integral*

$$\iint_S \mathbb{1}_D(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

If $\mathbb{1}_D$ is not integrable, then D does not have an area.

Example 5. *Let*

$$D = \{ (x, y) \in [0, 1] \times [0, 1] \mid x, y \in \mathbb{Q} \}.$$

Then D does not have an area.

Definition 6. *If $f \colon D \longrightarrow \mathbb{R}$ is a function and D is bounded, then pick $D \subset S \subset \mathbb{R}^2$ a rectangle. Define*

$$\tilde{f} \colon S \longrightarrow \mathbb{R},$$

by the rule

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{otherwise.} \end{cases}$$

*We say that f is **integrable** over D if $\tilde{f}$ is integrable over S. In this case*

$$\iint_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint_S \tilde{f}(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Proposition 7. *Let $D \subset \mathbb{R}^2$ be a bounded subset and let $f \colon D \longrightarrow \mathbb{R}$ and $g \colon D \longrightarrow \mathbb{R}$ be two integrable functions. Let λ be a scalar.*

Then

(1) *$f + g$ is integrable over D and*

$$\iint_D f(x, y) + g(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y + \iint_D g(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

(2) *λf is integrable over D and*

$$\iint_D \lambda f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \lambda \iint_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

(3) *If $f(x, y) \leq g(x, y)$ for any $(x, y) \in D$, then*

$$\iint_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y \leq \iint_D g(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

(4) *$|f|$ is integrable over D and*

$$| \iint_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y| \leq \iint_D |f(x, y)| \, \mathrm{d}x \, \mathrm{d}y.$$

It is straightforward to integrate continuous functions over regions of three special types:

Definition 8. *A bounded subset $D \subset \mathbb{R}^2$ is an **elementary region** if it is one of three types:*

Type 1:

$$D = \{ (x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \gamma(x) \leq y \leq \delta(x) \},$$

where $\gamma \colon [a, b] \longrightarrow \mathbb{R}$ and $\delta \colon [a, b] \longrightarrow \mathbb{R}$ are continuous functions.

Type 2:

$$D = \{ (x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \alpha(y) \leq x \leq \beta(y) \},$$

where $\alpha \colon [c, d] \longrightarrow \mathbb{R}$ and $\beta \colon [c, d] \longrightarrow \mathbb{R}$ are continuous functions.

Type 3: *D is both type 1 and 2.*

Theorem 9. *Let $D \subset \mathbb{R}^2$ be an elementary region and let $f \colon D \longrightarrow \mathbb{R}$ be a continuous function.*

Then

(1) *If D is of type 1, then*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\gamma(x)}^{\delta(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

(2) *If D if of type 2, then*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Example 10. *Let D be the region bounded by the lines $x = 0$, $y = 4$ and the parabola $y = x^2$. Let $f\colon D \longrightarrow \mathbb{R}$ be the function given by $f(x,y) = x^2 + y^2$.*

If we view D as a region of type 1, then we get

$$\begin{aligned}
\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \left(\int_{x^2}^4 x^2 + y^2\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^4 \mathrm{d}x \\
&= \int_0^2 4x^2 + \frac{2^6}{3} - x^4 - \frac{x^6}{3}\,\mathrm{d}x \\
&= \left[\frac{4x^3}{3} + \frac{2^6 x}{3} - \frac{x^5}{5} - \frac{x^7}{3\cdot 7} \right]_0^2 \\
&= \frac{2^5}{3} + \frac{2^7}{3} - \frac{2^5}{5} - \frac{2^7}{3\cdot 7} \\
&= \frac{2^6}{3\cdot 5} + \frac{2^8}{7} \\
&= 2^6 \left(\frac{1}{3\cdot 5} + \frac{2^2}{7} \right).
\end{aligned}$$

On the other hand, if we view D as a region of type 2, then we get

$$\begin{aligned}
\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^4 \left(\int_0^{\sqrt{y}} x^2 + y^2\,\mathrm{d}x \right) \mathrm{d}y \\
&= \int_0^4 \left[\frac{x^3}{3} + xy^2 \right]_0^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^4 \frac{y^{3/2}}{3} + y^{5/2}\,\mathrm{d}y \\
&= \left[\frac{2y^{5/2}}{3\cdot 5} + \frac{2y^{7/2}}{7} \right]_0^4 \\
&= \frac{2^6}{3\cdot 5} + \frac{2^8}{7} \\
&= 2^6 \left(\frac{1}{3\cdot 5} + \frac{2^2}{7} \right).
\end{aligned}$$