THE DOT PRODUCT
BASED ON LECTURE NOTES BY JAMES MCKERNAN

Blackboard 1. *Let $\vec{v}, \vec{w} \in \mathbb{R}^3$. Their **dot product** is*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Same in $\mathbb{R}^2$.

Example 2. *If $\vec{v} = (1, -2, -1)$ and $\vec{w} = (2, 1, -3)$ then*

$$\vec{v} \cdot \vec{w} = 1 \cdot 2 + (-2) \cdot 1 + (-1) \cdot (-3) = 2 - 2 + 3 = 3.$$

Lemma 3. *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and let $\lambda \in \mathbb{R}$.*

(1) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
(2) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
(3) $(\lambda \vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w})$.
(4) $\vec{v} \cdot \vec{v} = 0$ *if and only if* $\vec{v} = \vec{0}$.

Proof of (4).

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 = \|\vec{v}\|^2,$$

which is zero iff $\vec{v} = \vec{0}$. □

Blackboard 4. *Exercise: calculate $\vec{v} \cdot \vec{w}$ in terms of the norms of $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$.*

$$(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}$$
$$(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}.$$

Subtracting and dividing by 4 we get

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \frac{1}{4}\left((\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) - (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})\right) \\ &= \frac{1}{4}(\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2). \end{aligned}$$

Given two non-zero vectors $\vec{v}$ and $\vec{w}$ in space, note that we can define the angle θ between $\vec{v}$ and $\vec{w}$. $\vec{v}$ and $\vec{w}$ lie in at least one plane (which is in fact unique, unless $\vec{v}$ and $\vec{w}$ are parallel). Now just measure the angle θ between the $\vec{v}$ and $\vec{w}$ in this plane. By convention we always take $0 \leq \theta \leq \pi$.

Theorem 5. *If $\vec{v}$ and $\vec{w}$ are any two non-zero vectors in $\mathbb{R}^3$, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \; \|\vec{w}\| \cos\theta.$$

Proof. Apply the law of cosines to the triangle with sides parallel to $\vec{v}$, $\vec{w}$ and $\vec{v} - \vec{w}$:

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta.$$

Expand the LHS:

$$\|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta.$$

Canceling the common terms $\|\vec{v}\|^2$ and $\|\vec{w}\|^2$ from both sides, and dividing by -2, we get the desired formula. □

We can use (5) to find the angle between two vectors:

Example 6. *Let* $\vec{v} = -\hat{\imath} + \hat{k}$ *and* $\vec{w} = \hat{\imath} + \hat{\jmath}$. *Then*

$$-1 = \vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = 2\cos\theta.$$

Therefore $\cos\theta = -1/2$ *and so* $\theta = 2\pi/3$.

Blackboard 7. *We say that two vectors* $\vec{v}$ *and* $\vec{w}$ *in* $\mathbb{R}^3$ *are* ***orthogonal*** *if* $\vec{v}\cdot\vec{w} = 0$.

If neither $\vec{v}$ nor $\vec{w}$ are the zero vector, and $\vec{v} \cdot \vec{w} = 0$ then the angle between $\vec{v}$ and $\vec{w}$ is a right angle. By our convention, the zero vector is orthogonal to every vector.

Example 8. $\hat{\imath}$, $\hat{\jmath}$ *and* $\hat{k}$ *are pairwise orthogonal.*
(x, y, z) *is orthogonal to* $(y, -x, 0)$.

Given two vectors $\vec{v}$ and $\vec{w}$, we can project $\vec{v}$ onto $\vec{w}$. The resulting vector is called the **projection** of $\vec{v}$ onto $\vec{w}$.

Blackboard 9. *If* $\vec{w} \neq \vec{0}$ *then* $proj_{\vec{w}}\vec{v}$ *is the unique vector that is parallel to* $\vec{w}$ *and such that* $\vec{v} - proj_{\vec{w}}\vec{v}$ *is orthogonal to* $\vec{w}$.

For example, if $\vec{F}$ is a force and $\vec{w}$ is a direction, then the projection of $\vec{F}$ onto $\vec{w}$ is the force in the direction of $\vec{w}$.

Blackboard 10. *As* $proj_{\vec{w}}\vec{v}$ *is parallel to* $\vec{w}$, *we have*

$$proj_{\vec{w}}\vec{v} = \lambda\vec{w},$$

for some scalar λ. *Let's determine* λ. *Let's deal with the case that* $\lambda \geq 0$ *(so that the angle* θ *between* $\vec{v}$ *and* $\vec{w}$ *is between* 0 *and* $\pi/2$*). If we take the norm of both sides, we get*

$$\|proj_{\vec{w}}\vec{v}\| = \|\lambda\vec{w}\| = \lambda\|\vec{w}\|,$$

(note that $\lambda \geq 0$*), so that*

$$\lambda = \frac{\|proj_{\vec{w}}\vec{v}\|}{\|\vec{w}\|}.$$

But

$$\cos\theta = \frac{\|proj_{\vec{w}}\vec{v}\|}{\|\vec{v}\|},$$

so that

$$\|proj_{\vec{w}}\vec{v}\| = \|\vec{v}\|\cos\theta.$$

Putting all of this together we get

$$\begin{aligned}\lambda &= \frac{\|\vec{v}\|\cos\theta}{\|\vec{w}\|} \\ &= \frac{\|\vec{v}\|\|\vec{w}\|\cos\theta}{\|\vec{w}\|^2} \\ &= \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}.\end{aligned}$$

There are a number of ways to deal with the case when $\lambda < 0$ (so that $\theta > \pi/2$). One can carry out a similar analysis to the one given above. Here is another way.

Blackboard 11. *Note that the angle* ϕ *between* $\vec{w}$ *and* $\vec{u} = -\vec{v}$ *is equal to* $\pi - \theta < \pi/2$. *By what we already proved*

$$proj_{\vec{w}}\vec{u} = \frac{\vec{u} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}.$$

But $proj_{\vec{w}}\vec{u} = -proj_{\vec{w}}\vec{v}$ *and* $\vec{u} \cdot \vec{w} = -\vec{v} \cdot \vec{w}$, *so we get the same formula in the end.*

To summarize:

Theorem 12. *If* $\vec{v}$ *and* $\vec{w}$ *are two vectors in* $\mathbb{R}^3$, *where* $\vec{w}$ *is not zero, then*

$$proj_{\vec{w}}\vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w}.$$

Blackboard 13. *Recall that* $\hat{w} = \frac{1}{|\vec{w}|}\vec{w}$ *is the direction of* $\vec{w}$, *and that* $\alpha(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (\alpha\vec{w})$.

$$\begin{aligned} proj_{\vec{w}}\vec{v} &= \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w} \\ &= \left(\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}\right)\frac{\vec{w}}{|\vec{w}\|} \\ &= (\vec{v} \cdot \hat{w})\hat{w}. \end{aligned}$$

Example 14. *Find the distance* d *between the line* l *containing the points* $P_1 = (1, -1, 2)$ *and* $P_2 = (4, 1, 0)$ *and the point* $Q = (3, 2, 4)$.

Suppose that R *is the closest point on the line* l *to the point* Q. *Note that* $\overrightarrow{QR}$ *is orthogonal to the direction* $\overrightarrow{P_1P_2}$ *of the line. So we want the length of the vector* $\overrightarrow{P_1Q} - proj_{\overrightarrow{P_1P_2}}\overrightarrow{P_1Q}$.*, that is, we want*

$$d = \|\overrightarrow{P_1Q} - proj_{\overrightarrow{P_1P_2}}\overrightarrow{P_1Q}\|.$$

Now

$$\overrightarrow{P_1Q} = (2, 3, 2) \qquad \text{and} \qquad \overrightarrow{P_1P_2} = (3, 2, -2).$$

We have

$$\|\overrightarrow{P_1P_2}\|^2 = 3^2 + 2^2 + 2^2 = 17 \qquad \text{and} \qquad \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q} = 6 + 6 - 4 = 8.$$

It follows that

$$proj_{\overrightarrow{P_1P_2}}\overrightarrow{P_1Q} = \frac{8}{17}(3, 2, -2).$$

Subtracting, we get

$$\overrightarrow{P_1Q} - proj_{\overrightarrow{P_1P_2}}\overrightarrow{P_1Q} = (2, 3, 2) - \frac{8}{17}(3, 2, -2) = \frac{1}{17}(10, 35, 50) = \frac{5}{17}(2, 7, 10).$$

Taking the length, we get

$$\frac{5}{17}(2^2 + 7^2 + 10^2)^{1/2} \approx 3.64.$$

Theorem 15. *The angle subtended on the circumference of a circle by a diameter of the circle is always a right angle.*

Proof. Suppose that P and Q are the two endpoints of a diameter of the circle and that R is a point on the circumference. We want to show that the angle between $\overrightarrow{PR}$ and $\overrightarrow{QR}$ is a right angle.

Let O be the center of the circle. Then

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} \qquad \text{and} \qquad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}.$$

Note that $\overrightarrow{QO} = -\overrightarrow{PO}$. Therefore

$$\begin{aligned}\overrightarrow{PR} \cdot \overrightarrow{QR} &= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{QO} + \overrightarrow{OR}) \\ &= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{OR} - \overrightarrow{PO}) \\ &= \|\overrightarrow{OR}\|^2 - \|\overrightarrow{PO}\|^2 \\ &= r^2 - r^2 = 0,\end{aligned}$$

where r is the radius of the circle. It follows that $\overrightarrow{PR}$ and $\overrightarrow{QR}$ are indeed orthogonal. $\square$