Divergence, gradient and curl
Based on lecture notes by James McKernan

One can formally define the gradient of a function

$$\nabla f \colon \mathbb{R}^3 \longrightarrow \mathbb{R},$$

by the formal rule

$$\operatorname{grad} f = \nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Just like $\frac{d}{dx}$ is an operator that can be applied to a function, the del operator is a vector operator given by

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Using the operator del we can define two other operations, this time on vector fields:

Blackboard 1. *Let $A \subset \mathbb{R}^3$ be an open subset and let $\vec{F} \colon A \longrightarrow \mathbb{R}^3$ be a vector field.*

*The **divergence** of $\vec{F}$ is the scalar function,*

$$\operatorname{div} \vec{F} \colon A \longrightarrow \mathbb{R},$$

which is defined by the rule

$$\begin{aligned}
\operatorname{div} \vec{F}(x, y, z) &= \nabla \cdot \vec{F}(x, y, z) \\
&= \left(\hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \cdot (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.
\end{aligned}$$

*The **curl** of $\vec{F}$ is the vector field*

$$\operatorname{curl} \vec{F} \colon A \longrightarrow \mathbb{R}^3,$$

which is defined by the rule

$$\begin{aligned}
\operatorname{curl} \vec{F}(x, x, z) &= \nabla \times \vec{F}(x, y, z) \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.
\end{aligned}$$

Note that the del operator makes sense for any n, not just $n = 3$. So we can define the gradient and the divergence in all dimensions. However curl only makes sense when $n = 3$.

Blackboard 2. *The vector field $\vec{F} \colon A \longrightarrow \mathbb{R}^3$ is called **rotation free** if the curl is zero, $\operatorname{curl} \vec{F} = \vec{0}$, and it is called **incompressible** if the divergence is zero, $\operatorname{div} \vec{F} = 0$.*

Proposition 3. *Let f be a scalar field and $\vec{F}$ a vector field.*

(1) *If f is $\mathcal{C}^2$, then* $\operatorname{curl}(\operatorname{grad} f) = \vec{0}$. *Every conservative vector field is rotation free.*
(2) *If $\vec{F}$ is $\mathcal{C}^2$, then* $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$. *The curl of a vector field is incompressible.*

Proof. We compute;

$$\begin{aligned}
\operatorname{curl}(\operatorname{grad} f) &= \nabla \times (\nabla f) \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{\imath} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \hat{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \hat{k} \\
&= \vec{0}.
\end{aligned}$$

This gives (1).

$$\begin{aligned}
\operatorname{div}(\operatorname{curl} \vec{F}) &= \nabla \cdot (\nabla \times f) \\
&= \nabla \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\
&= 0.
\end{aligned}$$

This is (2). □

Example 4. *The gravitational field*

$$\vec{F}(x, y, z) = \frac{cx}{(x^2 + y^2 + z^2)^{3/2}} \hat{\imath} + \frac{cy}{(x^2 + y^2 + z^2)^{3/2}} \hat{\jmath} + \frac{cz}{(x^2 + y^2 + z^2)^{3/2}} \hat{k},$$

is a gradient vector field, so that the gravitational field is rotation free. In fact if

$$f(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}},$$

then $\vec{F} = \operatorname{grad} f$, *so that*

$$\operatorname{curl} \vec{F} = \operatorname{curl}(\operatorname{grad} f) = \vec{0}.$$

Example 5. *A magnetic field $\vec{B}$ is always the curl of something,*

$$\vec{B} = \operatorname{curl} \vec{A},$$

where $\vec{A}$ is a vector field. So

$$\operatorname{div}(\vec{B}) = \operatorname{div}(\operatorname{curl} \vec{A}) = 0.$$

Therefore a magnetic field is always incompressible.

There is one other way to combine two del operators:

Blackboard 6. *The **Laplace operator** take a scalar field $f\colon A \longrightarrow \mathbb{R}$ and outputs another scalar field*

$$\nabla^2 f\colon A \longrightarrow \mathbb{R}.$$

It is defined by the rule

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

A solution of the differential equation

$$\nabla^2 f = 0,$$

*is called a **harmonic function**.*

Example 7. *The function*

$$f(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}},$$

is harmonic.