Parametrised curves
Based on lecture notes by James McKernan

Blackboard 1. *A **parametrised differentiable curve in** $\mathbb{R}^n$ is a differentiable function $\vec{r}\colon I \longrightarrow \mathbb{R}^n$, where I is an open interval in $\mathbb{R}$.*

Remark 2. *Any open interval I is one of four different forms: (a, b); $(-\infty, b)$; (a, ∞); $(-\infty, \infty) = \mathbb{R}$, where a and b are real numbers.*

Blackboard 3. *The **velocity vector** at time t of a parametrised differentiable curve $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ is the derivative:*

$$\vec{v}(t) = \vec{r}'(t) = D\vec{r}(t).$$

*If $\vec{v}$ is differentiable, then the **acceleration vector** at time t is the derivative of the velocity vector:*

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

Example 4. *Let*

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^3,$$

be given by

$$\vec{r}(t) = (a\cos t, a\sin t, bt).$$

This traces out a helix. The velocity vector is

$$\vec{v}(t) = (-a\sin t, a\cos t, b).$$

The acceleration vector is

$$\vec{a}(t) = (-a\cos t, -a\sin t, 0).$$

The speed, that is the magnitude of the velocity vector,

$$\|\vec{v}(t)\| = (a^2 + b^2)^{1/2},$$

is constant. Nevertheless the acceleration vector is not zero, as we are travelling on a curve and not a straight line.

Blackboard 5. *Let $\vec{r}(t)$ be the position of a planet orbiting a sun. Kepler's second law: the area swept by $\vec{r}$ per unit of time is constant.*

Blackboard 6. *Newton's law of gravity: $\vec{a} = -\frac{GM}{|r(t)|^3}\vec{r}$.*

Claim 7. *The area swept by $\vec{r}$ per unit time is $\vec{r} \times \vec{v}$.*

Blackboard 8. *Proof of Kepler's second law, from Newton's law of gravity.*

$$\begin{aligned} \frac{d}{dt}(\vec{r} \times \vec{v}) &= \vec{r}' \times \vec{v} + \vec{r} \times \vec{v}' \\ &= \vec{v} \times \vec{v} + \vec{r} \times (-\frac{GM}{|r(t)|^3}\vec{r}) \\ &= 0. \end{aligned}$$

Hence $\vec{r}(t) \times \vec{v}(t) = \vec{c}$ is a constant, proving the second law. It also follows that $\vec{r}$ is always on the plane orthogonal to $\vec{c}$.

Note that Kepler's second law holds for any gravitational law for which $\vec{a}$ is proportional to $\vec{v}$.

Blackboard 9. *Let I be an open interval. A **partition** $\mathcal{P}$ of $[a,b] \subset I$ is a sequence of points*

$$a = t_0 < t_1 < \cdots < t_n = b,$$

*for some $n \geq 1$. The **mesh** of $\mathcal{P}$ is*

$$m(\mathcal{P}) = \max_i \{\, |t_{i+1} - t_i| \mid 0 \leq i \leq n-1 \,\}.$$

Blackboard 10. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a parametrised differentiable curve, and let $\mathcal{P}$ be a partition. We think of*

$$l(\mathcal{P}) = \sum_{i=0}^{n-1} \|\vec{r}(t_{i+1}) - r(t_i)\|,$$

as being an approximation to the length of the curve $\vec{r}([a,b])$.

*The curve $\vec{r}$ has **length** L if given any $\epsilon > 0$, there is a $\delta > 0$ such that for every partition $\mathcal{P}$ whose mesh size is less than δ, we have*

$$\|L - l(\mathcal{P})\| < \epsilon.$$

There are interesting examples of curves that don't have a length.

Start with interval $[0,1]$ in the plane. The length is 1. Now adjust this curve, by adding in a triangular hump in the middle, so that we get four line segments of length $1/3$. This is a curve of length $4/3$.

Now add a hump to each of the four line segments of length $1/3$. The length of the resulting curve is $(4/3)^2$. If we keep doing this, then we get more and more complicated curves, whose length at stage n is $(4/3)^n$. This process converges to a very pointed curve whose length is infinite (in fact this curve is a fractal).

However:

Proposition 11. *If $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^1$-function, then the curve $\vec{r}([a,b])$ has a length*

$$L = \int_a^b \|\vec{r}'(t)\| \, \mathrm{d}t.$$

Remark 12. *The fractal curve above is continuous but it is nowhere differentiable (the curve has too many sharp points).*

In general the exact formula for the arclength is only of theoretical interest. However there are some contrived examples where we can calculately the arclength precisely.

Example 13. *Let*

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

be the parametrised differentiable curve given by

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath}.$$

Then

$$\vec{r}'(t) = -a\sin t\hat{\imath} + a\cos t\hat{\jmath},$$

and so

$$\|\vec{r}'(t)\| = a.$$

Hence the length of the curve $\vec{r}([0,2\pi])$ is

$$L = \int_0^{2\pi} a \, \mathrm{d}t = 2\pi a,$$

which is indeed the circumference of a circle of radius a.

Example 14. *Let*

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^3,$$

be the parametrised differentiable curve given by

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath} + bt\hat{k},$$

Then

$$\vec{r}\,'(t) = -a\sin t\hat{\imath} + a\cos t\hat{\jmath} + b\hat{k},$$

and so

$$\|\vec{r}\,'(t)\| = \sqrt{a^2+b^2}.$$

Hence the length of the curve $\vec{r}([0,2\pi])$ is

$$L = \int_0^{2\pi} (a^2+b^2)^{1/2}\,\mathrm{d}t = 2\pi(a^2+b^2)^{1/2}.$$

Example 15. *Let*

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

be the parametrised differentiable curve given by

$$\vec{r}(t) = a\cos t\hat{\imath} + b\sin t\hat{\jmath}.$$

Then

$$\vec{r}\,'(t) = -a\sin t\hat{\imath} + b\cos t\hat{\jmath},$$

and so

$$\|\vec{r}\,'(t)\| = (a^2\sin^2 t + b^2\cos^2 t)^{1/2}$$

Hence the length of the curve $\vec{r}([0,2\pi])$ is

$$L = \int_0^{2\pi} (a^2\sin^2 t + b^2\cos^2 t)^{1/2}\,\mathrm{d}t,$$

the length of an ellipse, with major and minor axes of length a and b.

Blackboard 16. *We say a parametrised differentiable curve $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ is* ***regular*** *if $\vec{r}\,'(t) \neq 0$ (the speed is never zero).*

Blackboard 17. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a regular parametrised differentiable curve, which is of class $\mathcal{C}^1$. Given $a \in I$, define the* ***arclength parameter*** *$s(t)$, by the formula*

$$s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\,\mathrm{d}\tau$$

By the fundamental theorem of calculus

$$\begin{aligned} s'(t) &= \frac{d}{dt}\int_a^t \|\vec{r}\,'(\tau)\|\,\mathrm{d}\tau \\ &= \|\vec{r}\,'(t)\|, \end{aligned}$$

which is the speed at time t. Since $s'(t)$ is nowhere zero, we can write t as a function of s, that is we can write down the inverse function, $t(s)$, which will be $\mathcal{C}^1$.

Example 18. *For the helix,*

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath} + b\hat{k}t,$$

we have

$$s(t) = \int_0^t \|\vec{r}'(\tau)\|\,\mathrm{d}\tau = \sqrt{a^2+b^2}^{1/2}t.$$

Therefore

$$t(s) = \frac{s}{\sqrt{a^2+b^2}}.$$

In this case

$$\vec{r}(s) = a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\hat{\imath} + a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\hat{\jmath} + b\hat{k}\left(\frac{s}{\sqrt{a^2+b^2}}\right).$$

In fact, one can always parametrise a regular curve by its arclength and in this case the derivative is a unit vector:

Blackboard 19. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a parametrised differentiable curve parametrised by the arclength. Then*

$$\vec{T}(s) = \frac{d\vec{r}}{ds},$$

is the ***unit tangent vector****.*

To see that this is really a unit vector note that

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d\vec{r}}{ds}\frac{ds}{dt}\\ &= \vec{T}(s)\frac{ds}{dt},\end{aligned}$$

by the chain rule. So

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{\vec{r}'}{\|\vec{r}'\|}$$

so that the unit tangent vector is the unit vector which points in the direction of the velocity.

Example 20. *Let $\vec{r}(t) = (a\cos t, a\sin t)$ be the standard parametrisation of the circle of radius a. Then*

$$s = ta \qquad \textit{so that} \qquad t = \frac{s}{a}.$$

So

$$\vec{r}(s) = (a\cos\left(\frac{s}{a}\right), a\sin\left(\frac{s}{a}\right)),$$

and

$$\vec{T}(s) = (-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)).$$