

# UNIMODULARITY OF INVARIANT RANDOM SUBGROUPS

IAN BIRINGER AND OMER TAMUZ

ABSTRACT. An invariant random subgroup  $H \leq G$  is a random closed subgroup whose law is invariant to conjugation by all elements of  $G$ . When  $G$  is locally compact and second countable, we show that for every invariant random subgroup  $H \leq G$  there almost surely exists an invariant measure on  $G/H$ . Equivalently, the modular function of  $H$  is almost surely equal to the modular function of  $G$ , restricted to  $H$ .

We use this result to construct invariant measures on orbit equivalence relations of measure preserving actions. Additionally, we prove a mass transport principle for discrete or compact invariant random subgroups.

## 1. INTRODUCTION

Let  $G$  be a locally compact, second countable group. We denote by  $\text{Sub}_G$  the space of closed subgroups of  $G$ , equipped with the Chabauty topology, and consider the action  $G \curvearrowright \text{Sub}_G$  by conjugation.

An *invariant random subgroup* (IRS) is a  $\text{Sub}_G$ -valued random variable whose law is invariant to conjugation. Any Borel probability measure preserving action  $G \curvearrowright X$  gives an IRS - the stabilizer of a random point in  $X$ ; this follows<sup>1</sup> from the Compact Model Theorem of Varadarajan [24, Theorem 3.2 and its corollary]. In a slight (but standard) abuse of notation we also refer to a conjugation invariant Borel probability measure on  $\text{Sub}_G$  (i.e., the law of an IRS) as an IRS.

Although measure preserving  $G$ -actions and their stabilizers have been studied for some time (e.g., [9, 23]), IRSs were first introduced

---

*Date:* June 2, 2015.

*2010 Mathematics Subject Classification.* 28C10 (primary), 37A20 (secondary).

*Key words and phrases.* Invariant random subgroups, invariant measures on homogeneous spaces, mass transport principle.

<sup>1</sup>This theorem states that given a measurable action of a locally compact second countable group on a standard probability space  $X$ , one can endow  $X$  with a precompact metrizable topology under which the action is continuous. This implies that stabilizers are closed subgroups, and that the map that assigns to each point its stabilizer is upper semicontinuous and hence measurable.

by Abért, Glasner and Virág [3], and simultaneously by Vershik [26] under a different name. Since then, IRSs have appeared in a number of papers, either as direct subjects of study [12–14, 18, 26], as probabilistic limits of manifolds with increasing volume [1], or as tools to understand stationary group actions [11, 17].

The notion of an IRS is a natural weakening of that of a normal subgroup. As such, it is interesting to understand which properties of normal subgroups hold for IRSs. This is the spirit of [3], and of our main theorem.

More generally, one can consider the set of  $G$ -invariant,  $\sigma$ -finite Borel measures on  $\text{Sub}_G$ , which we denote by  $\mathcal{M}_G(\text{Sub}_G)$ . We will call an element of  $\mathcal{M}_G(\text{Sub}_G)$  an *invariant subgroup measure*. If  $(*)$  is a Borel property of subgroups of  $G$  (e.g. ‘discrete’ or ‘unimodular’) then we say that  $\lambda \in \mathcal{M}_G(\text{Sub}_G)$  is  $(*)$  whenever  $\lambda$ -a.e.  $H \in \text{Sub}_G$  is  $(*)$ .

Our main result is the following theorem.

**Theorem 1.1.** *Let  $\lambda \in \mathcal{M}_G(\text{Sub}_G)$  be an invariant subgroup measure. Then for all  $H$  in the topological support of  $\lambda$  there exists a nontrivial  $G$ -invariant measure on  $G/H$ .*

There exists a  $G$ -invariant measure on  $G/H$  if and only if the modular function  $\mu_G$  of  $G$  restricts to the modular function  $\mu_H$  of  $H$  (cf. [20, Theorem 1, pg 139]). Hence a corollary of Theorem 1.1 is that an invariant subgroup measure of a unimodular group  $G$  must be supported on unimodular subgroups.

When  $G$  is not unimodular, the kernel of the modular function is a unimodular, closed, normal proper subgroup of  $G$ . It can be thought of as the ‘unimodular radical’ of  $G$ : a normal unimodular subgroup that includes all other normal unimodular subgroups. Theorem 1.1 implies that any *unimodular* subgroup  $H \leq G$  that lies in the support of an invariant subgroup measure must be contained in  $\ker \mu_G$ . That is,

**Corollary 1.2.** *If an invariant subgroup measure  $\lambda \in \mathcal{M}_G(\text{Sub}_G)$  is unimodular, then  $H \leq \ker \mu_G$  for all  $H$  in the support of  $\lambda$ .*

Note that, in particular, this corollary applies whenever  $\lambda$  is supported on discrete subgroups. It would be interesting to understand when the same result can be proved for other ‘radicals’. For example, in [5] it is shown that every amenable IRS is included in the amenable radical.

Our proof of Theorem 1.1 yields a slightly stronger theorem (Theorem 3.1), in which the  $G$ -invariant measures on the coset spaces  $G/H$  can be chosen to vary measurably with  $H$ . As an application, we construct invariant measures on orbit equivalence relations.

Suppose that  $G \curvearrowright (X, \zeta)$  is a measure preserving action on a  $\sigma$ -finite Borel measure space. Let  $E \subseteq X \times X$  denote the orbit equivalence relation

$$E = \{(x, gx) : x \in X, g \in G\},$$

which we consider with the  $G$ -action  $g(x, y) = (x, gy)$ . Let  $p_l: E \rightarrow X$  be the projection on the left coordinate  $p_l(x, y) = x$ .

In many applications the interesting case is when  $\zeta$  is a probability measure. More generally, we prove:

**Corollary 1.3.** *Let  $G \curvearrowright (X, \zeta)$  be a measure preserving action on a  $\sigma$ -finite Borel measure space, with orbit equivalence relation  $E$ .*

*Then there exists a Borel family of  $G$ -invariant,  $\sigma$ -finite measures  $\nu_x$  on  $E$ , with  $\nu_x$  supported on the fiber  $p_l^{-1}(x)$ , and such that*

$$\nu = \int_X \nu_x d\zeta(x)$$

*is a  $G$ -invariant,  $\sigma$ -finite measure on  $E$ .*

When the action is free, each fiber  $p_l^{-1}(x)$  is identified with  $G$  and one can take  $\nu_x$  to be the Haar measure. When  $G$  is countable, the  $\nu_x$  are counting measures and Corollary 1.3 is due to Feldman and Moore [15, Theorem 2]. Such a measure  $\nu$  can be used, for example, to define the so called groupoid representation  $G \curvearrowright L^2(\nu)$  (see [21], and [25] for a recent application).

The final goal of this paper is to formulate a version of the ‘mass transport principle’ (MTP) for invariant subgroup measures. The MTP has proved to be a useful tool in the theory of random graphs [4, 8] and in percolation theory [6, 16]. Versions of the MTP have also appeared in the theory of foliations for some time (see, e.g., the definition of an invariant transverse measure in [19, page 82]).

A *mass transport* on a graph  $\Gamma = (V, E)$  is a positive function  $f: V \times V \rightarrow \mathbb{R}$ , where  $f(x, y)$  determines how much ‘‘mass’’ is transported from  $x$  to  $y$ . We assume that  $f$  is invariant to the diagonal action of the automorphism group  $\text{Aut}(\Gamma)$ , so that  $f(x, y)$  depends only on how  $x$  and  $y$  interact with the geometry of the graph. When  $\text{Aut}(\Gamma)$  is transitive and unimodular, a *mass transport principle* (MTP) applies: the total amount of mass transported from each vertex is equal to the amount transported to it. Namely, for all  $v \in V$ ,

$$(1.1) \quad \sum_{w \in V} f(v, w) = \sum_{w \in V} f(w, v).$$

The notion of a mass transport principle can be generalized to random rooted graphs. Let  $\mathcal{G}_\bullet$  be the space of isomorphism classes of

rooted graphs, equipped with the topology of convergence on finite neighborhoods. Similarly, let  $\mathcal{G}_{\bullet\bullet}$  be the space of isomorphism classes of doubly rooted graphs (see [8]). A random rooted graph ‘satisfies the mass transport principle’ when its law, a measure  $\lambda$  on  $\mathcal{G}_{\bullet}$ , satisfies

$$(1.2) \quad \int_{\mathcal{G}_{\bullet}} \sum_{w \in \Gamma} f(\Gamma, v, w) \, d\lambda(\Gamma, v) = \int_{\mathcal{G}_{\bullet}} \sum_{w \in \Gamma} f(\Gamma, w, v) \, d\lambda(\Gamma, v),$$

for all positive Borel functions  $f : \mathcal{G}_{\bullet\bullet} \rightarrow \mathbb{R}$ . Random rooted graphs that satisfy the MTP are also called *unimodular random graphs*. Let a random rooted graph  $\Gamma$  be almost surely a single transitive rooted graph. Then  $\Gamma$  is unimodular if and only if the automorphism group  $\text{Aut}(\Gamma)$  is a unimodular topological group, in which case (1.2) is exactly the MTP given in (1.1).

One should view an MTP as a replacement for group-invariance of a measure, which is especially useful when no group action is present. Our interest, however, lies with measures on  $\text{Sub}_G$ , where  $G$  acts by conjugation. We show that for measures supported on discrete or compact subgroups of  $G$ , conjugation invariance is equivalent to a suitable MTP. This generalizes the results of Abért, Glasner and Virág [3] for discrete  $G$ . We elaborate on this connection in Section 5. Discrete IRSs are particularly interesting in the case of Lie groups: the non-atomic IRSs of a connected simple Lie group are supported on discrete groups [1, Theorem 2.6].

Here, we will state our theorem in the case that  $G$  is unimodular, saving the more general statement for Section 5. So, suppose that  $G$  is a unimodular, locally compact, second countable group, and fix a Haar measure  $\ell$  on  $G$ . If  $H$  is a compact subgroup of  $G$ , let  $\nu_H$  be the push forward of  $\ell$  to  $H \backslash G$ , while if  $H$  is a discrete subgroup of  $G$ , let  $\nu_H$  be the measure on  $H \backslash G$  obtained by *locally pushing forward*  $\ell$  under the covering map  $G \rightarrow H \backslash G$ . This is the unique measure with

$$(1.3) \quad \ell = \int_{G/H} \eta_{Hg} \, d\nu_H(Hg),$$

where  $\eta_{Hg}$  is the counting measure on  $Hg$ . See Section 5 for details.

Denote by  $\text{Cos}_G$  the space of cosets of closed subgroups of  $G$ :

$$\text{Cos}_G = \{Hg : H \in \text{Sub}_G, g \in G\}.$$

We discuss this space and its topology in Section 2.

**Theorem 1.4** (Mass Transport Principle). *Let  $G$  be unimodular, and let  $\lambda$  be a  $\sigma$ -finite Borel measure on  $\text{Sub}_G$  such that  $\lambda$ -a.e.  $H \in \text{Sub}_G$  is discrete or compact.*

Then  $\lambda$  is conjugation invariant (i.e., an invariant subgroup measure) if and only if for every nonnegative Borel function  $f: \text{Cos}_G \rightarrow \mathbb{R}$ ,

$$\begin{aligned} & \int_{\text{Sub}_G} \int_{H \backslash G} f(Hg) \, d\nu_H(Hg) \, d\lambda(H) \\ &= \int_{\text{Sub}_G} \int_{H \backslash G} f(g^{-1}H) \, d\nu_H(Hg) \, d\lambda(H). \end{aligned}$$

Theorem 1.4 will be used by Abért and Biringer in [2] to show that certain invariant random subgroups of continuous groups correspond to ‘unimodular random manifolds’, i.e. measures on the space of rooted Riemannian manifolds satisfying a mass transport principle.

We should note that this is not the first version of a mass transport principle that applies in the continuous setting. In [7], Benjamini and Schramm give a version of the MTP for the hyperbolic plane.

To interpret Theorem 1.4, one should view  $\text{Cos}_G$  as foliated by the right coset spaces  $H \backslash G$ , where  $H$  ranges through  $\text{Sub}_G$ . The MTP says that the measure  $\nu$  obtained by integrating the measures  $\nu_H$  against  $\lambda$  is invariant under the involution  $Hg \mapsto g^{-1}H$  of  $\text{Cos}_G$ .

An alternative, appealing interpretation of the MTP is the following. Call a closed normal subgroup  $N \triangleleft G$  *co-unimodular* if  $G/N$  is unimodular; that is, if there exists a bi-invariant measure on  $G/N$ . Analogously, we call an IRS  $\lambda$  *co-unimodular* if there exists a Borel measure  $\nu$  on  $\text{Cos}_G$  that projects to  $\lambda$  and is invariant to both the left and right  $G$ -action. In Section 5.3 we show that when  $G$  is unimodular, then there is an MTP for  $\lambda$  if and only if  $\lambda$  is co-unimodular.

A particularly aesthetic version of the MTP arises if we also define for each discrete  $H \leq G$ , a measure  $\nu^H$  on  $G/H$  by locally pushing forward  $\ell$  with respect to the covering map  $G \rightarrow G/H$ . That is,

$$(1.4) \quad \ell = \int_{H \backslash G} \eta_{gH} \, d\nu^H(gH),$$

where  $\eta_{gH}$  is the counting measure on  $Hg$ . It follows from (1.3) and (1.4) that the involution  $Hg \mapsto g^{-1}H$  sends  $\nu_H$  to  $\nu^H$ . So, the MTP can be rephrased as

$$\begin{aligned} & \int_{\text{Sub}_G} \int_{H \backslash G} f(Hg) \, d\nu_H(Hg) \, d\lambda(H) \\ &= \int_{\text{Sub}_G} \int_{G/H} f(gH) \, d\nu^H(gH) \, d\lambda(H). \end{aligned}$$

In other words, the measure on  $\text{Cos}_G$  obtained by integrating the natural measures on right coset spaces  $H \backslash G$  against  $\lambda$  is the same as the

measure on  $\text{Cos}_G$  obtained by integrating the natural measures on left coset spaces  $G/H$  against  $\lambda$ .

**1.1. Acknowledgments.** The authors are indebted to Lewis Bowen, who first posed the question that led to Theorem 1.1, suggested that we include the statement of Theorem 1.3, and inspired the discussion in Remark 3.3. The first author is partially supported by NSF grant DMS-1308678 and would like to thank Miklos Abért for numerous conversations, in particular those relating to the mass transport principle, as without his input the statement given here might not have been considered or solved. The second author would like to thank Yair Hartman for enlightening discussions. Both authors would also like to thank the referee for greatly improving the readability and accuracy of the paper.

## 2. COSETS, SUBGROUPS AND A DISINTEGRATION RESULT

We start by defining the coset space of  $G$ ,  $\text{Cos}_G$ , which is the set

$$\text{Cos}_G = \{gH : g \in G, H \in \text{Sub}_G\},$$

equipped with the Fell topology of closed subsets of  $G$ . This topology is locally compact, second countable and Hausdorff, hence Polish. Note that since  $gH = gHg^{-1}g = H^g g$ , an equivalent definition is

$$\text{Cos}_G = \{Hg : g \in G, H \in \text{Sub}_G\}.$$

The group  $G$  acts on  $\text{Cos}_G$  from the left. When viewing  $\text{Cos}_G$  as the set of left cosets, the action of  $k \in G$  is given by  $k(gH) = kgH$ , and when considering right cosets we have  $k(Hg) = (kHk^{-1})(kg) = H^k kg$ .

Adopting the perspective of right cosets, consider the maps

$$\begin{array}{ccccc} \text{Sub}_G \times G & \xrightarrow{\sigma_r} & \text{Cos}_G & \xrightarrow{\pi_r} & \text{Sub}_G, \\ (H, g) & \mapsto & Hg & \mapsto & H. \end{array}$$

These maps are all  $G$ -equivariant, where the actions are  $(H, g) \xrightarrow{k} (kHk^{-1}, kg)$ ,  $Hg \xrightarrow{k} kHk^{-1}(kg)$ , and  $H \xrightarrow{k} kHk^{-1}$ . Note that  $\pi_r \circ \sigma_r$  is the projection  $(H, g) \mapsto H$ . We will also need the maps

$$\begin{array}{ccccc} \text{Sub}_G \times G & \xrightarrow{\sigma_l} & \text{Cos}_G & \xrightarrow{\pi_l} & \text{Sub}_G, \\ (H, g) & \mapsto & gH & \mapsto & H. \end{array}$$

When viewing  $\text{Cos}_G$  as right cosets,  $\pi_l(Hg) = \pi_l(g \cdot g^{-1}Hg) = g^{-1}Hg$ .

Suppose that  $G$  is a locally compact, second countable group. The main result of this section is the following disintegration theorem.

**Proposition 2.1.** *Fix a left Haar measure  $\ell$  on  $G$ , and a Borel map*

$$\text{Cos}_G \longrightarrow \mathcal{M}(G), \quad Hg \longmapsto \eta_{Hg}$$

*such that each  $\eta_{Hg}$  is nonzero, left  $H$ -invariant and supported on  $Hg$ .*

*Then for each  $H \in \text{Sub}_G$  there is a unique Borel measure  $\nu_H$  on  $H \backslash G \subset \text{Cos}_G$  such that*

$$\ell = \int_{H \backslash G} \eta_{Hg} \, d\nu_H(Hg).$$

*Moreover, the map  $\text{Sub}_G \longrightarrow \mathcal{M}(\text{Cos}_G)$ ,  $H \longmapsto \nu_H$  is Borel.*

Succinctly, the proposition states that left  $H$ -invariant measures on cosets  $Hg$  can be realized as fiber measures in a disintegration of any left Haar measure on  $G$ , and that if the fiber measures are Borel parametrized, so are the resulting factor measures.

*Proof.* Since  $\ell$  is  $\sigma$ -finite, the push forward measure on  $H \backslash G$  is equivalent to a  $\sigma$ -finite measure  $\hat{\nu}_H$ ; for instance, one can take  $\hat{\nu}_H$  to be the push forward of any probability measure on  $G$  that is equivalent to  $\ell$ . Note that the push forward of  $\ell$  is only itself  $\sigma$ -finite if  $H$  is compact.

Applying Rohlin's Disintegration Theorem (see [22, Theorem 6.3]), there is a disintegration

$$(2.1) \quad \ell = \int_{H \backslash G} \hat{\eta}_{Hg} \, d\hat{\nu}_H(Hg),$$

where  $Hg \longmapsto \hat{\eta}_{Hg}$  is a measurable parametrization of  $\sigma$ -finite measures on  $G$ , and  $\hat{\eta}_{Hg}$  is supported on the coset  $Hg$ .

We claim that  $\hat{\nu}_H$ -almost every  $\hat{\eta}_{Hg}$  is left  $H$ -invariant: left multiplication by  $h \in H$  on  $G$  leaves invariant each right coset  $Hg$ , and so

$$\ell = h_*(\ell) = \int_{H \backslash G} h_*(\hat{\eta}_{Hg}) \, d\hat{\nu}_H(Hg).$$

By the uniqueness of disintegrations with a given factor measure, we have  $h_*(\hat{\eta}_{Hg}) = \hat{\eta}_{Hg}$  for  $\hat{\nu}_H$ -a.e. coset  $Hg$ . It follows from the separability of  $H$  that  $\hat{\eta}_{Hg}$  is  $H$ -invariant for  $\hat{\nu}_H$ -a.e.  $Hg$ .

Equation (2.1) remains unchanged if we replace a  $\hat{\nu}_H$ -negligible number of the measures  $\hat{\eta}_{Hg}$  with  $\eta_{Hg}$ , so we may as well assume from now on that every  $\hat{\eta}_{Hg}$  is nonzero and left  $H$ -invariant. By the uniqueness of the Haar measure, each  $\hat{\eta}_{Hg}$  is a positive scalar multiple of  $\eta_{Hg}$ . Let

$$(2.2) \quad f : \text{Cos}_G \rightarrow \mathbb{R}^+, \quad f(Hg) = \frac{d\hat{\eta}_{Hg}}{d\eta_{Hg}},$$

be the function whose value gives this multiple. Define the measure  $\nu_H$  on  $H \backslash G$  by

$$d\nu_H(Hg) = f(Hg) d\hat{\nu}_H(Hg).$$

Then we have

$$\begin{aligned} \ell &= \int_{H \backslash G} \hat{\eta}_{Hg} d\hat{\nu}_H(Hg) \\ &= \int_{H \backslash G} \eta_{Hg} \cdot f(Hg) d\hat{\nu}_H(Hg) \\ (2.3) \quad &= \int_{H \backslash G} \eta_{Hg} d\nu_H(Hg), \end{aligned}$$

as required in the statement of the proposition. As the  $\eta_{Hg}$  are fixed,  $\nu_H$  is the unique measure on  $H \backslash G$  satisfying (2.3).

It remains to show that the  $\nu_H$  are Borel parametrized, when regarded as measures on the space of all cosets  $\text{Cos}_G$ . This is a consequence of the fact that the  $\eta_{Hg}$  are Borel parametrized, but we will never-the-less give a careful proof.

Fix a positive, continuous function  $F : G \rightarrow \mathbb{R}$  such that

$$\int F d\eta_{Hg} < \infty \quad \forall Hg \in \text{Cos}_G.$$

One way to produce such a function is as follows. Pick a compact neighborhood  $B$  of the identity, and a locally finite cover of  $G$  by open sets  $B_n$  with  $B_n^{-1}B_n \subset B$ . If  $hg \in Hg \cap B_n^{-1}$ , then

$$\begin{aligned} \eta_{Hg}(B_n) &= ((hg)_*^{-1} \eta_{Hg})(hgB_n) \\ &= \eta_{g^{-1}Hg}(hgB_n) \\ &< \eta_{g^{-1}Hg}(B). \end{aligned}$$

So, for each  $Hg$ , there is an upper bound on  $\eta_{Hg}(B_n)$  that is independent of  $n$ . The function  $F = \sum_n \frac{1}{2^n} \cdot \rho_n$ , where  $\rho_n$  is a partition of unity subordinate to the cover  $B_n$ , then has the desired properties.

Using  $F$ , we define a positive, Borel function

$$F' : \text{Sub}_G \times G \rightarrow \mathbb{R}, \quad F'(H, g) = \frac{F(g)}{\int F d\eta_{Hg}}.$$



By definition, this  $F'$  has the property that  $\int F'(H, \cdot) d\eta_{Hg} = 1$  for all  $Hg \in \text{Cos}_G$ . Given a continuous function  $\varphi : \text{Cos}_G \rightarrow \mathbb{R}$ , we have

$$\begin{aligned} \int \varphi(Hg) d\nu_H(Hg) &= \int \varphi(Hg) \left( \int_{k \in Hg} F'(H, k) d\eta_{Hg}(k) \right) d\nu_H(Hg) \\ &= \int \varphi(Hg) F'(H, g) d\ell(g). \end{aligned}$$

The last expression is Borel in  $H$ , so we have shown that integrating a fixed continuous function  $\varphi$  against the measures  $\nu_H$  gives a Borel function in  $H$ , implying that the map  $H \mapsto \nu_H$  is Borel.  $\square$

Finally, let us discuss a convenient construction of such  $\eta_{Hg}$ . In Claim A.2, we show that there is a continuous map

$$m : \text{Sub}_G \rightarrow \mathcal{M}(G)$$

such that each  $m(H)$  is a left Haar measure on  $H \subset G$ . Define

$$\eta_{Hg} := g_* m(g^{-1}Hg).$$

Each  $\eta_{Hg}$  is a  $\sigma$ -finite Borel measure supported on the coset  $Hg \subset G$ , and if  $h \in H$  we have

$$h_*(\eta_{Hg}) = h_* g_* m(g^{-1}Hg) = g_*(g^{-1}hg)_* m(g^{-1}Hg) = \eta_{Hg},$$

which shows that  $\eta_{Hg}$  is left  $H$ -invariant, and is well defined in the sense that it depends only on the coset  $Hg$  and not on the representative  $g$ . The  $\eta_{Hg}$  are also permuted by the left  $G$ -action: if  $k \in G$  then

$$k_*(\eta_{Hg}) = (kg)_* m(g^{-1}Hg) = \eta_{(kHk^{-1})kg} = \eta_{kHg}.$$

So, to summarize this construction:

**Fact 2.2.** *There is a family of measures  $\eta_{Hg}$  as required by Proposition 2.1 such that the map  $Hg \mapsto \eta_{Hg}$  is left  $G$ -equivariant:*

$$k_*(\eta_{Hg}) = \eta_{kHg}, \quad \forall k \in G, \quad Hg \in \text{Cos}_G.$$

Note that for some classes of subgroups  $H$ , there are ‘natural’ choices for a left Haar measure  $m(H)$ . For instance, if  $H$  is discrete, one can take  $m(H)$  to be the counting measure, in which case the measures  $\eta_{Hg}$  will all be counting measures as well. When  $H$  is compact, one can take  $m(H)$  to be the Haar probability measure.

As ‘discrete’ and ‘compact’ are both Borel properties of subgroups, the map  $H \mapsto m(H)$  above (and therefore the measures  $\eta_{Hg}$ ) can be adjusted to agree with these natural choices on such subgroups. This will be important in Section 5.

## 3. THE PROOF OF THEOREM 1.1

Our main result is the following theorem.

**Theorem 3.1.** *Let  $\lambda \in \mathcal{M}_G(\text{Sub}_G)$  be an invariant subgroup measure. Then there exists a Borel map*

$$\text{Sub}_G \longrightarrow \mathcal{M}(\text{Cos}_G), \quad H \longmapsto \nu^H$$

*such that for  $\lambda$ -a.e.  $H$  the measure  $\nu^H$  is nonzero,  $G$ -invariant and supported on the subset  $G/H \subset \text{Cos}_G$ .*

As we show in Lemma A.3, the set of all  $H \in \text{Sub}_G$  such that  $G/H$  as an invariant measure is closed. Therefore, we will obtain as a corollary of Theorem 3.1 our main theorem from the introduction:

**Theorem 1.1.** *Let  $\lambda \in \mathcal{M}_G(\text{Sub}_G)$  be an invariant subgroup measure. Then for all  $H$  in the topological support of  $\lambda$  there exists a nontrivial  $G$ -invariant measure on  $G/H$ .*

We devote the remainder of this section to proving Theorem 3.1, and so fix an invariant subgroup measure  $\lambda \in \mathcal{M}_G(\text{Sub}_G)$ .

**Proposition 3.2.** *There is a left  $G$ -invariant,  $\sigma$ -finite Borel measure  $\nu$  on  $\text{Cos}_G$  such that  $\pi_{l*}\nu \equiv \lambda \equiv \pi_{r*}\nu$ .*

Here, recall that  $\pi_l$  and  $\pi_r$  are the maps  $\text{Cos}_G \longrightarrow \text{Sub}_G$  taking  $gH \longmapsto H$  and  $Hg \longmapsto H$ , respectively.

Before proving Proposition 3.2 we show that it implies Theorem 3.1, and in fact is equivalent to it: Given Theorem 3.1, the measure  $\nu$  is obtained by integrating against  $\lambda$ :

$$(3.1) \quad \nu = \int_{\text{Sub}_G} \nu^H \, d\lambda(H).$$

This  $\nu$  is  $G$ -invariant since each  $\nu^H$  is  $G$ -invariant, and  $\pi_{l*}\nu \equiv \lambda$  since  $\nu^H$  is nonzero and supported on  $\pi_l^{-1}(H)$ .

Conversely, suppose Proposition 3.2 is true. Since  $\pi_{l*}\nu \equiv \lambda$ , Rohlin's Disintegration Theorem (see [22, Theorem 6.3]) implies that there is a Borel map  $H \longmapsto \nu^H \in \mathcal{M}(\text{Cos}_G)$ , with  $\nu^H$  supported on  $G/H = \pi_l^{-1}(H)$ , such that Equation (3.1) holds.

The action of  $G$  on  $\text{Cos}_G$  leaves invariant all fibers of the map  $\pi_l : \text{Cos}_G \longrightarrow \text{Sub}_G$ . So since  $\nu$  is  $G$ -invariant, the fiber measures  $\nu^H$  are  $G$ -invariant for  $\lambda$ -a.e.  $H \in \text{Sub}_G$ . In other words, for  $\lambda$ -a.e.  $H$ ,  $\nu^H$  is an invariant measure on  $G/H$ , proving Theorem 3.1.

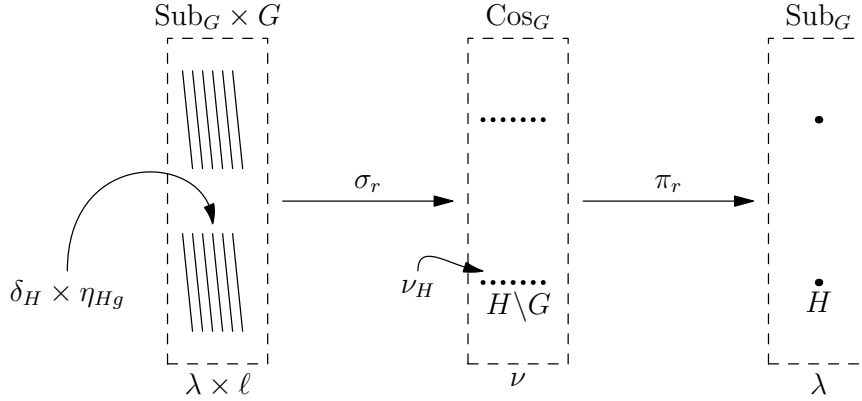


FIGURE 1. The measures involved in the proof of Proposition 3.2.

*Proof of Proposition 3.2.* Fix a left Haar measure  $\ell$  on  $G$ . Choosing a family  $\eta_{Hg}$  of measures as described in Fact 2.2, Proposition 2.1 gives a Borel family of measures  $\nu_H$  on  $H \backslash G \subset \text{Cos}_G$  with

$$\ell = \int_{H \backslash G} \eta_{Hg} d\nu_H(Hg)$$

for each  $H \in \text{Sub}_G$ , and we set

$$\nu = \int \nu_H d\lambda(H).$$

Note that  $\pi_{r*}\nu \equiv \lambda$ , since  $\nu_H$  is supported on the fiber  $\pi_r^{-1}(H)$ . The measure  $\lambda \times \ell$  on  $\text{Sub}_G \times G$  can then be expressed as

$$\begin{aligned} \lambda \times \ell &= \int_{\text{Sub}_G} \delta_H \times \left( \int_{H \backslash G} \eta_{Hg} d\nu_H(Hg) \right) d\lambda(H) \\ &= \int_{\text{Cos}_G} \delta_H \times \eta_{Hg} d\nu(Hg), \end{aligned}$$

where  $\delta_H$  is the Dirac measure at  $H \in \text{Sub}_G$ . This is a disintegration of  $\lambda \times \ell$  with respect to  $\sigma_r : \text{Sub}_G \times G \rightarrow \text{Cos}_G$ ,  $\sigma_r(H, g) = Hg$ .

The left  $G$ -action on  $\text{Sub}_G \times G$  is  $(H, g) \xrightarrow{k} (kHk^{-1}, kg)$ . Since  $\lambda$  is conjugation invariant and  $\ell$  is left  $G$ -invariant,  $\lambda \times \ell$  is left  $G$ -invariant. But by the equivariance in Fact 2.2, we have that if  $k \in G$ ,

$$k_*(\delta_H \times \eta_{Hg}) = \delta_{kHk^{-1}} \times \eta(kHk^{-1})(kg)$$

so the left  $G$ -action on  $\text{Sub}_G \times G$  permutes the fiber measures  $\delta_H \times \eta_{Hg}$ . Therefore, as  $\lambda \times \ell$  is invariant, and the collection of fiber measures of its disintegration is equivariant, we have that the factor measure  $\nu$  is invariant (c.f. Proposition B.1).

It remains to be shown that  $\pi_{l*}(\nu) \equiv \lambda$ . Let

$$\psi : \text{Sub}_G \times G \rightarrow \text{Sub}_G \times G$$

be given by  $\psi(H, g) = (H^{g^{-1}}, g)$ . Then

$$\pi_l \circ \sigma_r = \pi_r \circ \sigma_r \circ \psi,$$

since  $\pi_l \circ \sigma_r(H, g) = H^{g^{-1}}$  and  $\pi_r \circ \sigma_r$  is the projection on the first coordinate. But  $\psi$  preserves  $\lambda \times \ell$ , since  $\lambda$  is conjugation invariant, so

$$\pi_{l*}(\nu) \equiv (\pi_l \circ \sigma_r)_*(\lambda \times \ell) = (\pi_r \circ \sigma_r)_*(\lambda \times \ell) \equiv \lambda. \quad \square$$

**Remark 3.3.** *The measures  $\nu^H$  in Theorem 1.1 are only determined up to a positive multiple, so the integrated measure*

$$\nu = \int_{\text{Sub}_G} \nu^H d\lambda(H)$$

*is unique up to scaling by a positive Borel function  $f : \text{Cos}_G \rightarrow \mathbb{R}$  that is constant on each fiber  $\pi_l^{-1}(H) = G/H$ .*

*Sometimes, but not always, it is possible to choose the measure  $\nu$  in Proposition 3.2 to be invariant under both the left and right actions of  $G$  on  $\text{Cos}_G$ . If  $\lambda$  is ergodic, such a  $\nu$  is unique up to a global scalar. This is discussed in Section 5.3.*

#### 4. INVARIANT MEASURES ON ORBIT EQUIVALENCE RELATIONS

As in the introduction, suppose that  $G \curvearrowright (X, \zeta)$  is a measure preserving action on a standard Borel probability space and let

$$E = \{(x, gx) : x \in X, g \in G\} \subset X \times X$$

be the associated orbit equivalence relation, which we consider with the  $G$ -action  $g(x, y) = (x, gy)$  and the left projection  $p_l : E \rightarrow X$ ,  $p_l(x, y) = x$ . The standard fact that  $E$  is a Borel subset of  $X \times X$  follows (for example) from the Compact Model Theorem of Varadarajan [24, Theorem 3.2]; see for example [27, Corollary 2.1.20].

We aim to prove:

**Corollary 1.3.** *Let  $G \curvearrowright (X, \zeta)$  be a measure preserving action on a  $\sigma$ -finite Borel measure space, with orbit equivalence relation  $E$ .*

*Then there exists a Borel family of  $G$ -invariant,  $\sigma$ -finite measures  $\nu_x$  on  $E$ , with  $\nu_x$  supported on the fiber  $p_l^{-1}(x)$ , and such that*

$$\nu = \int_X \nu_x d\zeta(x)$$

*is a  $G$ -invariant,  $\sigma$ -finite measure on  $E$ .*

To begin the proof, note that  $E$  decomposes as a union of the fibers  $p_l^{-1}(x) = \{x\} \times Gx$ . The fiber  $\{x\} \times Gx$  is invariant under the  $G$ -action on  $E$ , and is isomorphic as a  $G$ -space to the quotient  $G/G_x$ , where

$$G_x = \{g \in G : gx = x\}$$

is the stabilizer of  $x$ . Let  $\text{stab} : X \rightarrow \text{Sub}_G$  be the map that assigns to each  $x \in X$  its stabilizer. As stabilizers of  $G$ -actions on separable Borel spaces are closed [24], the image of  $\text{stab}$  is in  $\text{Sub}_G$ . Since  $\text{stab}$  is equivariant,  $\text{stab}_*\zeta$  is an invariant subgroup measure. Theorem 3.1 then gives a Borel map

$$\text{Sub}_G \longrightarrow \mathcal{M}(\text{Cos}_G), \quad H \mapsto \nu^H$$

that associates to  $\text{stab}_*\zeta$ -a.e.  $H \in \text{Sub}_G$  a nonzero  $G$ -invariant measure on  $G/H \subset \text{Cos}_G$ . So, for  $\zeta$ -a.e.  $x \in X$  we can define  $\nu^x$  to be the measure on  $\{x\} \times Gx \cong G/G_x$  corresponding to  $\nu^{G_x}$ . The map

$$X \longrightarrow \mathcal{M}(E), \quad x \mapsto \nu^x$$

is then Borel, so we can integrate the measures  $\nu^x$  against the measure  $\zeta$  on  $X$  to give a  $\sigma$ -finite Borel measure  $\tau$  on  $E$ :

$$\tau = \int_X \nu^x \, d\zeta(x).$$

This  $\tau$  is invariant under the  $G$ -action on the second coordinate of  $E$ , since that action preserves the  $p_l$ -fibers  $\{x\} \times Gx$  and the associated measures  $\nu^x$ . Therefore, Corollary 1.3 follows.

## 5. THE MASS TRANSPORT PRINCIPLE

We start this section in 5.1 by an additional discussion of some well known aspects of mass transport principles and their relation to invariant random subgroups (cf. [3, 4]). We then prove our theorem in 5.2.

**5.1. The MTP for labeled graphs and discrete groups.** Fix a finite set  $S$ . An  $S$ -labeled graph is a countable directed graph with edges labeled by elements of  $S$ , such that the edges coming out from any given vertex  $v$  have labels in 1-1 correspondence with elements of  $S$ , and the same is true for the labels of edges coming into  $v$ . Let  $\mathcal{G}_\bullet^S$  and  $\mathcal{G}_{\bullet\bullet}^S$  be the spaces of rooted and doubly rooted  $S$ -labeled graphs, up to (label preserving) isomorphism. The topologies on  $\mathcal{G}_\bullet^S$  and  $\mathcal{G}_{\bullet\bullet}^S$  are defined so that two (doubly) rooted,  $S$ -labeled graphs are close when large finite balls around their roots are isomorphic.

A *unimodular random  $S$ -labeled graph* is a random  $S$ -labeled graph whose law is a probability measure  $\lambda$  on  $\mathcal{G}_\bullet^S$  such that for every non-negative Borel function  $f : \mathcal{G}_{\bullet\bullet}^S \rightarrow \mathbb{R}$  we have the *mass transport*

*principle*

$$(5.1) \quad \int_{\mathcal{G}_\bullet^S} \sum_{w \in G} f(G, v, w) \, d\lambda(G, v) = \int_{\mathcal{G}_\bullet^S} \sum_{w \in G} f(G, w, v) \, d\lambda(G, v).$$

In the introduction, we mentioned that unimodularity is often equivalent to invariance under a group action, when an action exists. Here, we can use the  $S$ -labels to construct an action of the free group  $F(S)$  generated by  $S$  on  $\mathcal{G}_\bullet^S$ , where the action of  $s$  moves the root along the adjacent *inward* edge labeled ‘ $s$ ’. We then have

**Fact 5.1.** *A random  $S$ -labeled graph is unimodular if and only if it is invariant under the action of  $F(S)$ .*

*Proof.* Assume first that  $\lambda$  is the law of a unimodular random  $S$ -labeled graph. If  $s \in S$  and  $E \subset \mathcal{G}_\bullet^S$  is Borel, define  $f : \mathcal{G}_{\bullet\bullet}^S \rightarrow \mathbb{R}$  by

$$f(G, v, w) = \begin{cases} 1_E(G, v) & (G, v) = s(G, w) \\ 0 & \text{otherwise} \end{cases}$$

Then the left side of the MTP is  $\lambda(E)$ , while the right-hand side is  $\lambda(s(E))$ . Therefore,  $\lambda$  is  $s$ -invariant.

For the other direction, suppose that  $\lambda$  is invariant under the action of each  $s \in S$ . By a standard reduction, it suffices to prove the MTP for functions  $f : \mathcal{G}_{\bullet\bullet}^S \rightarrow \mathbb{R}$  supported on graphs  $(G, v, w)$  where there is a directed edge from  $w$  to  $v$  (compare with Proposition 2.2 in [4]). If  $f$  is such a function, the left side of the MTP becomes

$$(5.2) \quad \int_{\mathcal{G}_\bullet^S} \sum_{s \in S \cup S^{-1}} \frac{f(G, v, sv)}{m(G, x, s)} \, d\lambda(G, v).$$

Here,  $S^{-1}$  is the set of formal inverses of elements of  $S$ , where the action of  $s^{-1}$  moves a vertex of a graph along the adjacent *outward* edge labeled ‘ $s$ ’. The multiplicity function  $m(G, x, s)$  records the number of elements  $t \in S \cup S^{-1}$  where  $sx = tx$ . Then

$$\begin{aligned} (5.2) &= \sum_{s \in S \cup S^{-1}} \int_{\mathcal{G}_\bullet^S} \frac{f(G, v, sv)}{m(G, v, s)} \, d\lambda(G, v) \\ &= \sum_{s \in S \cup S^{-1}} \int_{\mathcal{G}_\bullet^S} \frac{f(G, s^{-1}v, v)}{m(G, s^{-1}v, s)} \, d\lambda(G, v) \\ &= \int_{\mathcal{G}_\bullet^S} \sum_{s \in S \cup S^{-1}} \frac{f(G, sv, v)}{m(G, sv, s^{-1})} \, d\lambda(G, v) \\ &= \int_{\mathcal{G}_\bullet^S} \sum_{w \in G} f(G, w, v) \, d\lambda(G, v), \end{aligned}$$

which proves the mass transport principle.  $\square$

In [3], Abért, Glasner and Virág show how to produce unimodular random  $S$ -labeled graphs from invariant subgroup measures. Suppose that  $G$  is a group generated by the finite set  $S$ . The *Schreier graph* of a subgroup  $H \leq G$  is the graph  $\text{Sch}(H \backslash G, S)$  whose vertices are right cosets of  $H$  and where each  $s \in S$  contributes a directed edge labeled ‘s’ from every coset  $Hg$  to  $Hgs$ . We consider  $\text{Sch}(H \backslash G, S)$  as an  $S$ -labeled graph rooted at the identity coset  $H$ , in which case the action of  $F(S)$  described above is  $Hg \xrightarrow{s} Hgs^{-1}$ . This defines an injection

$$\Phi : \text{Sub}_G \longrightarrow \mathcal{G}_\bullet^S, \quad H \longmapsto (\text{Sch}(H \backslash G, S), H).$$

Under  $\Phi$ , the conjugation action of an element  $s \in S$  on  $\text{Sub}_G$  corresponds to the natural action on  $\mathcal{G}_\bullet^S$ , since

$$(\text{Sch}(H \backslash G, S), Hs^{-1}) \cong (\text{Sch}(sHs^{-1} \backslash G, S), sHs^{-1})$$

as  $S$ -labeled rooted graphs. Therefore, invariant subgroup measures of  $G$  produce  $F(S)$ -invariant measures on  $\mathcal{G}_\bullet^S$ .

We can also use the map  $\Phi$  to reinterpret the MTP for  $S$ -labeled graphs (5.1) group theoretically. Let  $\text{Cos}_G$  be the space of cosets of subgroups of  $G$ , as in Section 3. There is then an injection

$$(5.3) \quad \Phi' : \text{Cos}_G \longrightarrow \mathcal{G}_{\bullet\bullet}^S, \quad Hg \longmapsto (\text{Sch}(H \backslash G, S), H, Hg).$$

To interpret the right hand side of (5.1), note that the doubly rooted graph  $(\text{Sch}(H \backslash G, S), Hg, H)$  that arises from switching the order of the roots in  $\Phi(H)$  is isomorphic as an  $S$ -labeled graph to

$$(\text{Sch}(g^{-1}Hg \backslash G, S), g^{-1}Hg, g^{-1}Hgg^{-1}) = \Phi'((g^{-1}Hg)g^{-1}) = \Phi'(g^{-1}H).$$

Therefore, Fact 5.1 translates under  $\Phi$  and  $\Phi'$  to the following characterization of invariant subgroup measures of a discrete group  $G$ .

**The Discrete MTP.** *Suppose that  $G$  is a finitely generated group and  $\lambda$  is a Borel measure on  $\text{Sub}_G$ . Then  $\lambda$  is conjugation invariant if and only if for every nonnegative Borel function  $f : \text{Cos}_G \longrightarrow \mathbb{R}$ ,*

$$\int_{\text{Sub}_G} \sum_{H \backslash G} f(Hg) \, d\lambda(H) = \int_{\text{Sub}_G} \sum_{H \backslash G} f(g^{-1}H) \, d\lambda(H).$$

Re-indexing, a slightly more aesthetic statement of the discrete MTP is obtained by replacing the sum on the right with  $\sum_{G/H} f(gH)$ .

**5.2. Proof of the Mass Transport Theorem.** We now extend the Discrete MTP to general  $G$  and discrete or compact  $\lambda$ . As the following is intimately related to the existence of an invariant measure on  $\text{Cos}_G$ , we will frequently reference the setup of Sections 2 and 3.

Suppose that  $G$  is a locally compact, second countable topological group, and fix a left invariant Haar measure  $\ell$  on  $G$ . If  $H$  is a compact subgroup of  $G$ , let  $\nu_H$  be the push forward of  $\ell$  to  $H \backslash G$ . If  $H$  is a discrete subgroup of  $G$ , let  $\nu_H$  be the measure on  $H \backslash G$  obtained by *locally* pushing forward  $\ell$  under the covering map  $G \rightarrow H \backslash G$ . That is, if  $U \subset H \backslash G$  is an evenly covered open set with preimage  $V_1 \sqcup V_2 \sqcup \dots$ , then  $\nu_H|_U$  is the push forward of  $\ell|_{V_i}$  for every  $i$ .

These  $\nu_H$  can be understood in terms of Proposition 2.1. For discrete cosets, set  $\eta_{Hg}$  to be the counting measure. When  $H$  is compact, let  $\eta_{Hg}$  be the unique left  $H$ -invariant probability measure on  $Hg$ . Then in both cases,  $\nu_H$  is characterized by the equation

$$(5.4) \quad \ell = \int_{H \backslash G} \eta_{Hg} d\nu_H.$$

Note that by Proposition 2.1, the map  $H \mapsto \nu_H$  from  $\text{Sub}_G$  to the space of measures on  $\text{Cos}_G$  is Borel.

**Theorem 5.2** (Mass Transport Principle). *Let  $\lambda$  be a  $\sigma$ -finite Borel measure on  $\text{Sub}_G$  such that  $\lambda$ -a.e.  $H \in \text{Sub}_G$  is discrete or compact.*

*Then  $\lambda$  is conjugation invariant (i.e., an invariant subgroup measure) if and only if  $\mu_G|_H = 1$  for  $\lambda$ -a.e.  $H \in \text{Sub}_G$ , and for every nonnegative Borel function  $f: \text{Cos}_G \rightarrow \mathbb{R}$ ,*

$$(5.5) \quad \begin{aligned} & \int_{\text{Sub}_G} \int_{H \backslash G} f(Hg) \mu_G(Hg) d\nu_H(Hg) d\lambda(H) \\ &= \int_{\text{Sub}_G} \int_{H \backslash G} f(g^{-1}H) d\nu_H(Hg) d\lambda(H). \end{aligned}$$

Here,  $\mu_G$  is the modular function of  $G$ , defined by the equation

$$\ell(S) = \mu_G(g)\ell(Sg), \quad \text{for every Borel } S \subset G.$$

The assumption  $\mu_G|_H = 1$  implies that  $\mu_G$  is constant over each coset  $Hg$ , and we write  $\mu_G(Hg)$  for the common value on that coset. Note that the modular function of a discrete or compact group is trivial, so Theorem 1.1 implies that if  $\lambda$  is a discrete or compact invariant subgroup measure, then  $\mu_G|_H = \mu_H = 1$  automatically for  $\lambda$ -a.e.  $H$ . So, in the course of the proof below, we will always assume  $\mu_G|_H = \mu_H = 1$ .



The MTP can be stated in a way that is more similar to our work earlier in the paper. Define an involution

$$\rho: \text{Cos}_G \longrightarrow \text{Cos}_G, \quad \rho(Hg) = g^{-1}H,$$

and let  $\nu$  be the measure on  $\text{Cos}_G$  defined by the integral

$$\nu := \int_{\text{Sub}_G} \nu_H d\lambda(H).$$

Note that the map  $Hg \mapsto \eta_{Hg}$  defined above Equation (5.4) is left  $G$ -equivariant, so just as in the proof of Proposition 3.2 the measure  $\nu$  is left  $G$ -invariant. Now, changing variables on the right hand side, Equation (5.5) can be rewritten as:

$$(5.6) \quad \int_{\text{Cos}_G} f(Hg)\mu(Hg) d\nu(Hg) = \int_{\text{Cos}_G} f(Hg) d\rho_*\nu(Hg).$$

So, the MTP reduces to the following claim.

**Claim 5.3.**  *$\lambda$  is conjugation invariant if and only if  $\frac{1}{\mu_G} \cdot \nu = \rho_*\nu$ .*

*Proof of Claim 5.3.* Recall the following maps defined in Section 2:

$$\begin{array}{ccc} \text{Sub}_G \times G & \xrightarrow{\sigma_r} & \text{Cos}_G & \xrightarrow{\pi_r} & \text{Sub}_G, \\ (H, g) & \mapsto & Hg & \mapsto & H. \end{array}$$

As in the proof of Proposition 3.2, we have a  $\sigma_r$ -disintegration

$$(5.7) \quad \lambda \times \ell = \int_{\text{Cos}_G} \delta_H \times \eta_{Hg} d\nu(Hg),$$

where  $\ell$  is our chosen left Haar measure on  $G$ . Since  $\rho(Hg) = g^{-1}H = Hg^{-1}g^{-1}$ , there is a commutative diagram

$$\begin{array}{ccc} \text{Sub}_G \times G & \xrightarrow{\varphi} & \text{Sub}_G \times G, \\ \downarrow \sigma_r & & \downarrow \sigma_r \\ \text{Cos}_G & \xrightarrow{\rho} & \text{Cos}_G \end{array}$$

where  $\varphi$  is defined by  $\varphi(H, g) = (Hg^{-1}, g^{-1})$ . Note that if

$$\iota: G \longrightarrow G, \quad \iota(g) = g^{-1},$$

then  $\iota_*(\ell) = \frac{1}{\mu_G} \cdot \ell$ , so as  $\varphi$  inverts the second factor in  $\text{Sub}_G \times G$  and conjugates the first,  $\lambda$  is conjugation invariant if and only if

$$(5.8) \quad \varphi_*(\lambda \times \ell)(H, g) = \frac{1}{\mu_G(g)} \lambda \times \ell(H, g).$$

For each  $Hg \in \text{Cos}_G$ , the measure  $\eta_{Hg}$  is either the counting measure or the unique  $H$ -invariant probability measure on  $Hg$ . In addition to

being left  $H$ -invariant, in both cases  $\eta_{Hg}$  is in fact invariant under the right action of  $H^{g^{-1}}$  on  $Hg$ . This is immediate when it is a counting measure, and when  $H$  is compact,  $\eta_{Hg}$  is the push forward under left multiplication by  $g$  of the *bi-invariant* Haar probability measure on the compact group  $H^{g^{-1}}$ . So, the pushforward of  $\eta_{Hg}$  under inversion  $g \mapsto g^{-1}$  is a *left*  $H^{g^{-1}}$ -invariant measure on  $H^{g^{-1}}g^{-1}$ , and must be  $\eta_{H^{g^{-1}}g^{-1}}$ . Multiplying by  $\delta_H$ , we then have

$$\varphi_*(\delta_H \times \eta_{Hg}) = \delta_{H^{g^{-1}}} \times \eta_{H^{g^{-1}}g^{-1}}.$$

So, the fiber measures of the disintegration in (5.7) are permuted by  $\varphi$ . From the commutative diagram for  $\varphi$  and  $\rho$ , it follows that  $\rho$  scales  $\nu$  by  $1/\mu_G$  if and only if  $\varphi$  scales  $\lambda \times \ell$  by  $1/\mu_G$ , which we saw above was equivalent to conjugation invariance of  $\lambda$ .  $\square$

**5.3. Bi-invariant measures and co-unimodular IRSs.** As a consequence of Claim 5.3, we record the following.

**Corollary 5.4.** *Suppose that  $\lambda$  is an invariant subgroup measure in a unimodular group  $G$  such that  $\lambda$ -a.e.  $H \in \text{Sub}_G$  is discrete or cocompact. Then there is a measure  $\nu$  on  $\text{Cos}_G$  that is invariant under both the left and right actions of  $G$ , and for which  $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$ . Moreover, if  $\lambda$  is ergodic then  $\nu$  is unique up to scale.*

*Proof.* The construction above gives a measure  $\nu$  on  $\text{Cos}_G$  that is left  $G$ -invariant and also  $\rho$ -invariant. But the map  $\rho(Hg) = g^{-1}H$  conjugates the left  $G$ -action on  $\text{Cos}_G$  to the right  $G$ -action, so  $\nu$  is also right  $G$ -invariant. The fact that  $\pi_{r*}(\nu) \equiv \lambda$  is just the definition of  $\nu$ , and the fact that  $\pi_{l*}(\nu) \equiv \lambda$  is the argument at the end of Proposition 3.2.

Since  $\nu$  is bi-invariant, it is preserved under conjugation by  $k \in G$ , i.e. by the map

$$\text{Cos}_G \longrightarrow \text{Cos}_G, \quad gH \xrightarrow{k} kgHk = kgk^{-1}(kHk^{-1}).$$

This map sends  $G/H$  to  $G/(kHk^{-1})$ , so permutes the fibers of  $\pi_l$ . As it also preserves  $\nu$  and its  $\pi_l$ -factor measure  $\lambda$ , the conjugation map must permute the fiber measures  $\nu^H$ . Therefore, if  $\nu$  and  $\nu'$  are both bi-invariant and  $\lambda$  is ergodic, the function

$$H \longrightarrow \frac{d\nu^H}{d(\nu')^H}$$

is conjugation invariant, so is constant on a  $\lambda$ -full measure set. This implies that  $\nu$  and  $\nu'$  agree up to a scalar multiple, as desired.  $\square$

Bi-invariant  $\nu$  do not always exist for general invariant subgroup measures  $\lambda$ , even in unimodular groups  $G$ . For instance, if  $\lambda$  is an

atomic measure on a normal subgroup  $N \triangleleft G$ , then  $\nu = \nu_N$  is just the left Haar measure on  $G/N \subset \text{Cos}_G$ , which is right invariant exactly when  $G/N$  is unimodular. Note that unimodular  $G$  may have non-unimodular quotients  $G/N$ : an example is

$$\text{Sol} = \mathbb{R}^2 \rtimes \mathbb{R}, \quad t \in \mathbb{R} \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \circlearrowleft \mathbb{R}^2,$$

where  $N$  is the  $x$ -axis in  $\mathbb{R}^2 \subset \text{Sol}$ . Although  $\text{Sol}$  is unimodular, the quotient  $\text{Sol}/N$  is the group of affine transformations

$$\text{Aff}(\mathbb{R}) = \{x \mapsto ax + b \mid a, b \in \mathbb{R}\},$$

which is not unimodular.

As suggested in the introduction, it is natural to call an IRS (or more generally, an invariant subgroup measure)  $\lambda$  co-unimodular if there exists a bi-invariant  $\nu$  such that  $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$ . Using this terminology, Corollary 5.4 states that inside unimodular  $G$ , discrete IRSs are co-unimodular, as are compact ones. In general, co-unimodularity is equivalent to obeying a mass transport principle that is not twisted by the modular function.

**Claim 5.5.** *Let  $\lambda$  be an invariant subgroup measure in a locally compact second countable group  $G$ . Then the following are equivalent.*

- (1) *There exists a  $G$ -left-invariant measure  $\nu$  on  $\text{Cos}_G$  such that  $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$  and such that  $\rho_*\nu = \nu$  (i.e. an untwisted MTP holds for  $\nu$ ; see (5.6) without the  $\mu_G$  factor).*
- (2) *There exists a  $G$ -bi-invariant measure  $\nu$  on  $\text{Cos}_G$  such that  $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$  (that is,  $\lambda$  is co-unimodular).*

*Proof.* Fix a left-invariant measure  $\nu$  such that  $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$ . If  $\rho_*\nu = \nu$  then  $\nu$  is bi-invariant, following the argument of Corollary 5.4. Conversely, if  $\nu$  is bi-invariant, then  $\nu + \rho_*\nu$  has the same properties of  $\nu$ , but is also  $\rho$ -invariant.  $\square$

## APPENDIX A. SOME NOTES ON HAAR MEASURES

Suppose that  $G$  is a locally compact, second countable group and fix throughout this section a continuous, nonnegative function  $f : G \rightarrow \mathbb{R}$  with compact support such that  $f(1) = 1$ .

**Definition A.1.** If  $H$  is a closed subgroup of  $G$ , we define  $m_f(H)$  to be the unique left Haar measure on  $H$  such that  $\int f dm_f(H) = 1$ .

The measure  $m_f(H)$  will exist and be unique as long as we have that the integral of  $f$  against any left Haar measure  $\mu$  is nonzero and finite. The former follows since  $\mu$  is positive on open subsets of  $H$  and  $f$  is

positive on a neighborhood of the identity. The latter holds since  $\mu$  is Radon and  $\text{supp}(f|_H)$  is compact.

**Claim A.2.** *The map  $m_f : \text{Sub}_G \rightarrow \mathcal{M}(G)$  is continuous, where  $\text{Sub}_G$  has the Chabauty topology and  $\mathcal{M}(G)$  the topology of weak convergence.*

*Proof.* Let  $H_i \rightarrow H$  in  $\text{Sub}_G$  and set  $m_i = m_f(H_i)$ . Fix any subsequence  $(m_{i_j})$  of  $m_i$ . By [10, Proposition 2], the space of Haar measures on closed subgroups of  $G$  with the normalization  $\int f dm_f(H) = 1$  is weak\* compact. Therefore,  $(m_{i_j})$  has a further subsequence that converges to some Haar measure  $m$  for a closed subgroup of  $G$ .

We claim that  $m$  is  $H$ -invariant. Let  $h \in H$  and take a sequence of elements  $h_{i_j} \in H_{i_j}$  with  $h_{i_j} \rightarrow h$ . Then for any continuous function  $g : G \rightarrow \mathbb{R}$  with compact support, we have

$$\begin{aligned} \int g(hx) dm(x) &= \lim_j \int g(h_{i_j}x) dm_{i_j}(x) \\ &= \lim_j \int g(x) dm_{i_j}(x) \\ &= \int g(x) dm(x). \end{aligned}$$

The justification for the double limit is that  $g(h_{i_j} \cdot) \rightarrow g(h \cdot)$  converge uniformly and all have support within some compact  $K$ , on which we have  $m_{i_j}(K)$  bounded independently of  $j$ . It follows that  $m$  is  $H$ -invariant.

We now claim that  $m$  is supported within  $H$ . For given  $g \in G \setminus H$ , let  $U$  be an open set that is disjoint from some neighborhood of  $H$ . By the definition of the Chabauty topology,  $U \cap H_{i_j} = \emptyset$  for large  $j$ . Therefore,  $m_{i_j}(U) = 0$  for large  $j$ , implying that  $m(U) = 0$ .

This shows that  $m$  is a left Haar measure on  $H$ . Finally, as

$$\int f dm = \lim_j \int f dm_{i_j} = \lim_j 1 = 1,$$

it must be that  $m = m_f(H)$ , so the claim follows.  $\square$

**Lemma A.3.** *Suppose that  $G$  is a locally compact topological group. Then the subset  $\mathcal{U} \subset \text{Sub}_G$  consisting of subgroups  $H$  for which  $G/H$  has an invariant measure is closed.*

*Proof.* The space  $G/H$  admits a  $G$ -invariant measure if and only if the modular function of  $G$  restricts to the modular function of  $H$ : that is,  $\mu_G(h) = \mu_H(h)$  for all  $h \in H$  (see, e.g., [20]). So, suppose that we have a sequence of elements  $H_i \in \mathcal{U}$  and that  $H_i \rightarrow H$ . If  $h \in H$ , we want

to show that  $\mu_G(h) = \mu_H(h)$ . Let  $h_i \in H_i$  with  $h_i \rightarrow h$ . Then

$$\begin{aligned} h_* m_f(H) &= \lim_i (h_i)_* m_f(H_i) \\ &= \lim_i \mu_{H_i}(h_i) m_f(H_i) \\ &= \lim_i \mu_G(h_i) m_f(H_i) \\ &= \mu_G(h) m_f(H), \end{aligned}$$

so  $\mu_G(h) = \mu_H(h)$  and the lemma follows.  $\square$

## APPENDIX B. UNIQUENESS OF FACTOR MEASURES

We give here a proof of the following standard uniqueness statement for factor measures in a disintegration with prescribed non-zero fiber measures.

**Proposition B.1.** *Let  $X$  and  $Y$  be Borel spaces,  $p : X \rightarrow Y$  be a Borel map and*

$$Y \rightarrow \mathcal{M}(X), \quad y \mapsto \eta_y$$

*be a Borel parametrization of a family of  $\sigma$ -finite Borel measures on  $X$  such that each  $\eta_y$  is nonzero and supported on  $p^{-1}(y)$ . If  $\mu$  and  $\mu'$  are  $\sigma$ -finite Borel measures on  $Y$ , let*

$$\lambda = \int_Y \eta_y d\mu(y), \quad \lambda' = \int_Y \eta_y d\mu'(y).$$

*Then  $\lambda = \lambda'$  if and only if  $\mu = \mu'$ .*

*Proof.* The backwards implication is immediate. So, assume  $\lambda = \lambda'$ . We claim that there is a Borel function  $f : X \rightarrow \mathbb{R}$  such that

$$\int_X f d\eta_y = 1$$

for both  $\mu$ -a.e. and  $\mu'$ -a.e.  $y \in Y$ . This will finish the proof, since if  $U \subset Y$  is Borel then

$$\mu(U) = \int_X f \cdot 1_U(p(x)) d\lambda(x) = \int_X f \cdot 1_U(p(x)) d\lambda'(x) = \mu'(U).$$

The measure  $\lambda$  is  $\sigma$ -finite, so let  $U_1 \subset U_2 \subset \dots$  be an increasing sequence of Borel subsets of  $X$  with  $\lambda(U_i) < \infty$  and  $\cup_i U_i = X$ . For each  $y \in Y$ , let  $n(y)$  be the minimum  $i$  such that  $\eta_y(U_i) > 0$ . Define

$$f : X \rightarrow \mathbb{R}, \quad f(x) = \sum_{y \in Y} \frac{1_{U_{n(y)} \cap p^{-1}(y)}(x)}{\eta_y(U_{n(y)})}.$$

Then  $f$  is Borel, and the claim will follow if we show that  $\eta_y(U_{n(y)}) < \infty$  for both  $\mu$ -a.e. and  $\mu'$ -a.e.  $y \in Y$ . Assume by contradiction that there

exists a  $V \subseteq Y$  with  $\eta_y(U_{n(y)}) = \infty$  for all  $y \in V$ , and, say,  $\mu(V) > 0$ . It follows that there exists a  $W \subseteq V$ , with  $\mu(W) > 0$ , and an  $N$  such that  $\eta_y(U_N) = \infty$  for all  $y \in W$ . But then

$$\lambda(U_N) = \int_Y \eta_y(U_N) \, d\mu \geq \int_W \eta_y(U_N) \, d\mu = \infty,$$

which contradicts the initial choice of the sets  $U_i$ .  $\square$

## REFERENCES

- [1] Miklós Abért, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet, *On the growth of  $L^2$ -invariants for sequences of lattices in lie groups*, arXiv preprint arXiv:1210.2961 (2012).
- [2] Miklós Abért and Ian Biringer, *Invariant random hyperbolic 3-manifolds with finitely generated  $\pi_1$  are doubly degenerate*, 2014. Unpublished draft.
- [3] Miklós Abért, Yair Glasner, and Bálint Virág, *Kesten’s theorem for invariant random subgroups*, arXiv preprint arXiv:1201.3399 (2012).
- [4] David Aldous and Russell Lyons, *Processes on unimodular random networks*, Electron. J. Probab. **12** (2007), no. 54, 1454–1508. MR2354165 (2008m:60012)
- [5] Uri Bader, Bruno Duchesne, and Jean Lécureux, *Amenable invariant random subgroups*, arXiv preprint arXiv:1409.4745 (2014).
- [6] Itai Benjamini, Russell Lyons, Yuval Peres, and Oded Schramm, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), no. 1, 29–66. MR1675890 (99m:60149)
- [7] Itai Benjamini and Oded Schramm, *Percolation in the hyperbolic plane*, J. Amer. Math. Soc. **14** (2001), no. 2, 487–507 (electronic). MR1815220 (2002h:82049)
- [8] ———, *Recurrence of distributional limits of finite planar graphs*, Electron. J. Probab. **6** (2001), no. 23, 13 pp. (electronic). MR1873300 (2002m:82025)
- [9] Nicolas Bergeron and Damien Gaboriau, *Asymptotique des nombres de Betti,  $\ell^2$ -invariants et laminations*, Commentarii Mathematici Helvetici **79** (2004), no. 2, 362–395.
- [10] Nicolas Bourbaki, *Integration. II. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004. Translated from the 1963 and 1969 French originals by Sterling K. Berberian. MR2098271 (2005f:28001)
- [11] Lewis Bowen, *Random walks on coset spaces with applications to Furstenberg entropy*, Arxiv preprint arXiv:1008.4933 (2010).
- [12] ———, *Invariant random subgroups of the free group*, arXiv preprint arXiv:1204.5939 (2012).
- [13] Lewis Bowen, Rostislav Grigorchuk, and Rostyslav Kravchenko, *Invariant random subgroups of the lamplighter group*, arXiv preprint arXiv:1206.6780 (2012).
- [14] Darren Creutz and Jesse Peterson, *Stabilizers of ergodic actions of lattices and commensurators*, arXiv preprint arXiv:1303.3949 (2013).
- [15] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*, Bull. Amer. Math. Soc. **81** (1975), no. 5, 921–924. MR0425075 (54 #13033)

- [16] Olle Häggström, *Infinite clusters in dependent automorphism invariant percolation on trees*, Ann. Probab. **25** (1997), no. 3, 1423–1436. MR1457624 (98f:60207)
- [17] Yair Hartman and Omer Tamuz, *Furstenberg entropy realizations for virtually free groups and lamplighter groups*, arXiv preprint arXiv:1210.5897 (2012).
- [18] ———, *Stabilizer rigidity in irreducible group actions*, arXiv preprint arXiv:1307.7539 (2013).
- [19] Calvin C. Moore and Claude L. Schochet, *Global analysis on foliated spaces*, Second, Mathematical Sciences Research Institute Publications, vol. 9, Cambridge University Press, New York, 2006. MR2202625 (2006i:58035)
- [20] Leopoldo Nachbin, *The Haar integral*, Van Nostrand, 1965.
- [21] Jean Renault, *A groupoid approach to  $C^*$ -algebras*, Lecture notes in mathematics, Springer-Verlag, Berlin, 1980.
- [22] David Simmons, *Conditional measures and conditional expectation; Rohlin's disintegration theorem*, Discrete Contin. Dyn. Syst. **32** (2012), no. 7, 2565–2582. MR2900561
- [23] Garrett Stuck and Robert J. Zimmer, *Stabilizers for ergodic actions of higher rank semisimple groups*, Annals of Mathematics (1994), 723–747.
- [24] Veeravalli S. Varadarajan, *Groups of automorphisms of borel spaces*, Transactions of the American Mathematical Society **109** (1963), no. 2, 191–220.
- [25] Anatolii Moiseevich Vershik, *Nonfree actions of countable groups and their characters*, Journal of Mathematical Sciences **174** (2011), no. 1, 1–6.
- [26] ———, *Totally nonfree actions and the infinite symmetric group*, Moscow Mathematical Journal **12** (2012), no. 1, 193–212.
- [27] Robert J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1984.

(I. Biringer) DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL MA, 02467, USA.

(O. Tamuz) MICROSOFT RESEARCH NEW ENGLAND, ONE MEMORIAL DRIVE, CAMBRIDGE MA, 02142, USA.